Joint Statistics for Two Correlated Weibull Variates

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***Abstract*—This paper derives closed-form formulas for the joint statistics of two correlated Weibull variates. In particular, the joint probability density function, the general joint moments, and the general correlation coefficient are obtained in terms of well-known fading physical parameters.**

***Index Terms*—Correlated Weibull, Weibull correlation coefficient, Weibull distribution.**

I. Introduction

The Weibull distribution is an empirical distribution, which was originally used as a statistical model for reliability analysis. Its simplicity and flexibility soon paved its way to wireless communications applications, as widely reported in the literature (e.g., [1]–[6]). Of course, due to its origin, no physical model concerning the fading environment has been associated with it. In [7], a physical model for a generalized fading distribution was proposed, in which Weibull appears as a special case. In essence, as proposed in [7], the fading model for the Weibull distribution considers a signal composed of a cluster of multipath waves propagating in a nonhomogeneous environment. Within this cluster, the phases of the scattered waves are random. The resulting envelope is obtained as a nonlinear function of the modulus of the sum of the multipath components. Such a nonlinearity is manifested in terms of a power parameter, so that the resulting signal intensity is obtained not simply as the modulus of the sum of the multipath components, but as this modulus to a certain given exponent. The author of [7] does not try to explain why or how such a nonlinearity occurs or even if it indeed occurs. What the author conjectures about is that the resulting effect on the received signal propagated in a certain medium is manifested in terms of a nonlinearity. Besides the phenomenon related to the propagation medium, such a nonlinearity might also account for the practical limitations of the detection process at the receiver. The aim of the present paper is to derive closed-form formulas for the joint statistics of two correlated Weibull variates. In particular, the joint probability density function (pdf), the general joint moments, and the general correlation coefficient are obtained in terms of well-known fading physical parameters.

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II. Related Works

In [8], using a result of [9], a Weibull bivariate distribution is used to investigate the performance of dual selection diversity in correlated fading channels. In [9], the bivariate distribution is derived as a mixture of two Weibull marginals. The combination of the marginals is carried out in terms of a "dependence parameter" λ, which is not the correlation coefficient δ of two correlated Weibull variates.[1] The mathematical relation $\delta = f(\lambda)$ was found by using the statistical definition of correlation coefficient. The resulting equation was such that the dependence parameter could not be expressed directly in terms of the correlation coefficient, i.e., $\lambda = f^{-1}(\delta)$ could not be found explicitly. Therefore, the bivariate distribution, which is written in terms of the dependence parameter λ, appears only *indirectly* in terms of the correlation coefficient δ of its variates. Both λ and δ are basically nondimensional variables ranging from zero to one, and they bear no relation to physical parameters affecting the fading phenomena. One evidence substantiated by the authors of [8] was that, in spite of the fact that Weibull includes Rayleigh as a special case, the resulting joint Weibull pdf of [8] does not comprise the joint Rayleigh pdf [10]–[13] as a particular case. In this paper, a *simple* joint Weibull pdf is obtained in which the correlation coefficient appears *explicitly* in the resulting equation. Such a correlation coefficient is obtained in terms of well-known physical fading parameters. Therefore, all of the joint statistics can be written as functions of these fading parameters. In addition, the joint Rayleigh pdf constitutes a special case of the proposed distribution. It is important to mention that both pdfs, namely that of [8] as well as the one to be presented here, are legitimate joint Weibull pdfs, the validity of which in wireless communications can only be attested with practical field experimentation. There are, however, some important attributes concerning the joint Weibull pdf as proposed here: it is simple; it is fully characterized in terms of physical fading parameters; it is consistent with the other more general joint pdfs used in wireless communications, such as Rice [10], [11] and Nakagami-m [12], since it also encompasses the joint Rayleigh pdf as a particular case.

III. Preliminaries

In accordance with [7], the resulting signal envelope of a Weibull process is a nonlinear process obtained not simply as the modulus of the sum of the multipath components, but as this modulus to a certain given exponent. Suppose that such a nonlinearity is in the form of a power parameter $\alpha > 0$ so that the resulting envelope R is given by

$$R^{\alpha} = X_i^2 + Y_i^2 \tag{1}$$

[1] In [8], the dependence parameter and the correlation coefficient were named δ and ρ, respectively.

where X_i and Y_i are mutually independent Gaussian processes, with $E(X_i) = E(Y_i) = 0$ and $E(X_i^2) = E(Y_i^2) = \hat{r}^\alpha/2$, and $E(\cdot)$ is the expectation operator. From (1), it can be shown that the pdf $f_R(r)$ of R is found as

$$f_R(r) = \frac{\alpha r^{\alpha-1}}{\hat{r}^\alpha} \exp\left(-\frac{r^\alpha}{\hat{r}^\alpha}\right) \tag{2}$$

which is Weibull. The parameter $\hat{r}$, as defined here, is the α-root mean value of R^α, i.e., $\hat{r} = \sqrt[\alpha]{E(R^\alpha)}$. For a normalized envelope $\mathrm{P} = R/\hat{r}$, the pdf $f_\mathrm{P}(\rho)$ of P is obtained as

$$f_\mathrm{P}(\rho) = \frac{\alpha \rho^{\alpha-1}}{\exp(\rho^\alpha)}. \tag{3}$$

The probability distribution function $F_Z(z)$ of a Weibull variate Z can be found in a closed-form formula. In particular, $F_R(r)$ for the envelope R is given by

$$F_R(r) = 1 - \exp\left(-\frac{r^\alpha}{\hat{r}^\alpha}\right). \tag{4}$$

Equivalently

$$F_\mathrm{P}(\rho) = 1 - \exp(-\rho^\alpha). \tag{5}$$

The kth moment $E(\mathrm{P}^k)$ is obtained as

$$E(\mathrm{P}^k) = \Gamma(1 + k/\alpha) \tag{6}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, dt$ is the gamma function. Of course, $E(R^k) = \hat{r}^k E(\mathrm{P}^k)$. From (1), it can be seen that

$$R^\alpha = R_R^2 \tag{7}$$

where R_R is the Rayleigh envelope. The joint statistics can now be obtained by capitalizing on some results already available in the literature for the Rayleigh distribution. More specifically, we use the relation given in (7) between the Weibull and Rayleigh variates. Let R_{R1} and R_{R2} be two Rayleigh variates whose marginal statistics are respectively described by the parameters $E(R_{R1}^2) = \Omega_1$ and $E(R_{R2}^2) = \Omega_2$; R_1 and R_2 be two Weibull variates whose marginal statistics are respectively described by the parameters $\alpha_1, \hat{r}_1$ and $\alpha_2, \hat{r}_2$; and $0 \le \delta \le 1$ be a correlation parameter. (We postpone the discussion about this parameter to a later subsection.)

IV. Joint Probability Density Function

The joint pdf $f_{R_{R1},R_{R2}}(r_{R1}, r_{R2})$ of two Rayleigh variates with marginal statistics as described previously is given by (122) of [12] (or, equivalently, by (3.7-13) of [11]). By means of (7), so that $R_1^{\alpha_1} = R_{R1}^2$ and $R_2^{\alpha_2} = R_{R2}^2$, we find that $\hat{r}_1^{\alpha_1} = \Omega_1$ and $\hat{r}_2^{\alpha_2} = \Omega_2$. Now with (122) of [12] and the relations just given, the joint pdf $f_{R_1,R_2}(r_1, r_2)$ of two Weibull variates is found as $f_{R_1,R_2}(r_1, r_2) = |J| f_{R_{R1},R_{R2}}(r_{R1}, r_{R2})$, in which J is the Jacobian of the transformation. Following the standard statistical procedure of transformation of variates and after some algebraic manipulations, the joint pdf $f_{\mathrm{P}_1,\mathrm{P}_2}(\rho_1, \rho_2)$ of the Weibull variates $\mathrm{P}_1 = R_1/\hat{r}_1$ and $\mathrm{P}_2 = R_2/\hat{r}_2$ is found as

$$f_{\mathrm{P}_1,\mathrm{P}_2}(\rho_1, \rho_2) = \frac{\alpha_1 \alpha_2 \rho_1^{\alpha_1-1} \rho_2^{\alpha_2-1}}{(1-\delta)} \times \exp\left(-\frac{\rho_1^{\alpha_1} + \rho_2^{\alpha_2}}{1-\delta}\right) I_0\left(\frac{2\sqrt{\delta \rho_1^{\alpha_1} \rho_2^{\alpha_2}}}{1-\delta}\right) \tag{8}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind and order ν ([14], 9.6.18). For $\alpha_1 = \alpha_2 = 2$, then (8) reduces to (122) of [12], i.e., for this condition, (8) yields the joint distribution of two Rayleigh envelopes, as desired.

V. Generalized Joint Moments

The joint moments $E(\mathrm{P}_1^p \mathrm{P}_2^q)$ of two normalized Weibull variates may be found from (8) by using the standard procedure in probability theory (i.e., $E(\mathrm{P}_1^p \mathrm{P}_2^q) = \int_0^\infty \int_0^\infty \rho_1^p \rho_1^q f_{\mathrm{P}_1,\mathrm{P}_1}(\rho_1, \rho_2)\, d\rho_1\, d\rho_2$). Therefore, after several algebraic manipulations

$$E\left(\mathrm{P}_1^p \mathrm{P}_2^q\right) = \Gamma\left(1 + \frac{p}{\alpha_1}\right) \Gamma\left(1 + \frac{q}{\alpha_2}\right) {}_2F_1\left(-\frac{p}{\alpha_1}, -\frac{q}{\alpha_2}; 1; \delta\right) \tag{9}$$

where ${}_2F_1(,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function ([14], 15.1.1). Of course, $E(R_1^p R_2^q) = \hat{r}_1^p \hat{r}_2^q E(\mathrm{P}_1^p \mathrm{P}_2^q)$. For $\alpha_1 = \alpha_2 = 2$, then (9) reduces to (137) of [12] with $m = 1$, i.e., for this condition, (9) yields the generalized joint moments of two Rayleigh envelopes.

VI. Generalized Correlation Coefficient

Define a generalized correlation coefficient $\delta_{p,q}$ of two Weibull variates such that

$$\delta_{p,q} = \frac{C\left(R_1^p, R_2^q\right)}{\sqrt{V\left(R_1^p\right) V\left(R_2^q\right)}} = \frac{C\left(\mathrm{P}_1^p, \mathrm{P}_2^q\right)}{\sqrt{V\left(\mathrm{P}_1^p\right) V\left(\mathrm{P}_2^q\right)}} \tag{10}$$

where $V(\cdot)$ and $C(\cdot,\cdot)$ are the variance and covariance operators, respectively. By means of (6) and (9), and after some algebraic manipulations. (Please see (11) at the bottom of the page.) For $\alpha_1 = \alpha_2 = 2$, then (11) yields the generalized correlation coefficient of two Rayleigh envelopes. For $\alpha_1 = \alpha_2 = 2$ and $p = q$, then (11) reduces to (139) of [12] with $m = 1$.

$$\delta_{p,q} = \frac{\Gamma\left(1 + \frac{p}{\alpha_1}\right) \Gamma\left(1 + \frac{q}{\alpha_2}\right) \left({}_2F_1\left(-\frac{p}{\alpha_1}, -\frac{q}{\alpha_2}; 1; \delta\right) - 1\right)}{\sqrt{\left(\Gamma\left(1 + \frac{2p}{\alpha_1}\right) - \Gamma^2\left(1 + \frac{p}{\alpha_1}\right)\right) \left(\Gamma\left(1 + \frac{2q}{\alpha_2}\right) - \Gamma^2\left(1 + \frac{q}{\alpha_2}\right)\right)}} \tag{11}$$

VII. Some Insight Into the Correlation Coefficient

Let $\delta^R_{p,q}$ denote the generalized correlation coefficient of the Rayleigh distribution such that

$$\delta^R_{p,q} = \frac{C\left(R^p_{R1}, R^q_{R2}\right)}{\sqrt{V\left(R^p_{R1}\right) V\left(R^q_{R2}\right)}}. \tag{12}$$

Then, using (10), (12), and the relation as in (7) it can be seen that $\delta_{\alpha_1,\alpha_2} = \delta^R_{2,2}$. Interestingly, from (11), we find that $\delta_{\alpha_1,\alpha_2} = \delta$. This is a remarkable result, which shows that the correlation coefficient δ equals the correlation coefficient of the Weibull distribution for $p = \alpha_1$ and $q = \alpha_2$, and this is identical to the correlation coefficient of two squared Rayleigh envelopes. Therefore

$$\delta = \delta_{\alpha_1,\alpha_2} = \delta^R_{2,2}. \tag{13}$$

Note the that the Weibull correlation coefficient $\delta_{p,q}$ can now be expressed in terms of the Rayleigh correlation coefficient $\delta^R_{2,2}$. We then write the Rayleigh processes in terms of the Gaussian in-phase and quadrature processes as $R^2_{\text{Ri}} = X_i^2 + Y_i^2, i = 1, 2$, in which X_i, Y_i are zero-mean Gaussian in-phase and quadrature components. For the Rayleigh process $E(X_iX_j) = E(Y_iY_j), \forall i, j, E(X_iY_j) = -E(X_jY_i), i \neq j$. For any two Gaussian processes $G_i, i = 1, 2$, for which $E(G_i) = 0$, then $E(G_i^4) = 3E(G_i^2)$ and $E(G_1^2G_2^2) = E(G_1^2)E(G_2^2) + 2E^2(G_1G_2)$. Using these in (13) and after algebraic manipulations

$$\delta = \frac{E^2(X_1X_2) + E^2(X_1Y_2)}{E^2(X_1^2)}. \tag{14}$$

Any fading model in which the above statistics (14) are known can be used in order to obtain $\delta = \delta^R_{2,2}$. Therefore, the Weibull correlation coefficient $\delta_{p,q}$ can be expressed in terms of joint statistics of the in-phase and quadrature components of the Rayleigh process. In particular, for the Jakes model [13], we use (1.5-11), (1.5-14), and (1.5-15) of [13] such that (Please see (15) at the bottom of the page.) where: $D(\theta)$ is the horizontal directivity pattern of the receiving antenna; Θ is a variate denoting the angle of the incident power; ω is the maximum Doppler shift; τ is the time difference between the two fading signals; $\Delta\omega$ is the frequency difference between these signals; and T is a variate denoting the time delay. For an isotropic scattering (i.e., uniform distribution in angle of the incident power), omni-directional receiving antenna $(D(\theta) = 1)$, and exponentially distributed time delay [13]

$$\delta = \frac{J_0^2(\omega\tau)}{1 + (\Delta\omega t)^2} \tag{16}$$

in which t is the delay spread.

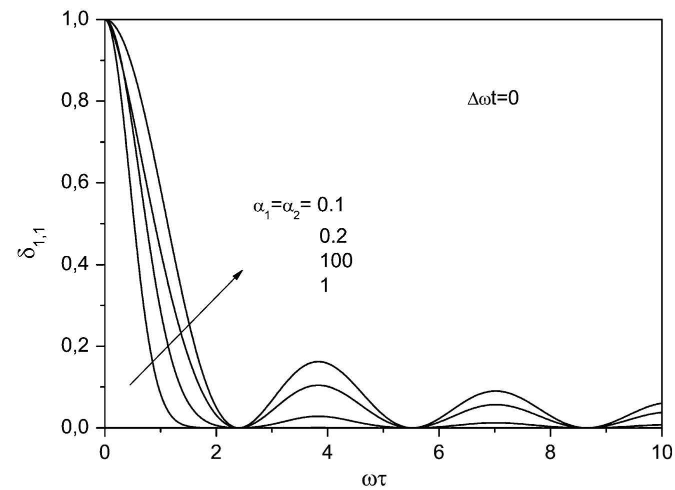

Fig. 1. The correlation coefficient for the Weibull envelopes as a function of $\omega\tau$.

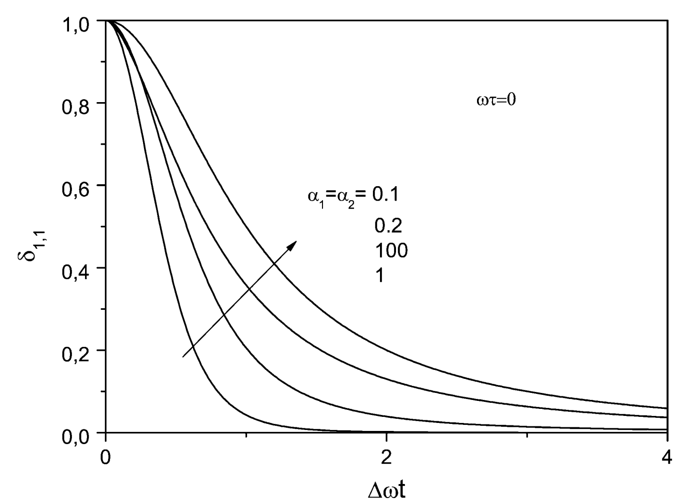

Fig. 2. The correlation coefficient for the Weibull envelopes as a function of $\Delta\omega t$.

VIII. Sample Examples

This section illustrates how the correlation coefficient of the Weibull envelopes varies for the isotropic condition. In this case, (16) is used into (11) and $p = q = 1$. Fig. 1 depicts $\delta_{p,q}$ as a function of $\omega\tau$ for different values of the Weibull parameters $\alpha_1 = \alpha_2$ and $\Delta\omega t = 0$. Fig. 2 shows $\delta_{p,q}$ as a function of $\Delta\omega t$ for different values of the Weibull parameters $\alpha_1 = \alpha_2$ and $\omega\tau = 0$. Note, in Figs. 1 and 2, that for $\alpha_1 = \alpha_2 > 1$, a large variation of the Weibull parameter, namely from $\alpha_1 = \alpha_2 = 1$ to $\alpha_1 = \alpha_2 = 100$, implies a small variation in the curves. In fact, it has been observed that the curve for which $\alpha_1 = \alpha_2 = 100$ is practically coincident with that for which $\alpha_1 = \alpha_2 \to \infty$. Therefore, we conclude that the correlation coefficient does not vary much for $\alpha_1 = \alpha_2 > 1$. The same does

$$\delta = \frac{E^2(D(\Theta)\cos(\omega\tau\cos\Theta - \Delta\omega T)) + E^2(D(\Theta)\sin(\omega\tau\cos\Theta - \Delta\omega T))}{E^2(D(\Theta))} \tag{15}$$

not hold for $\alpha_1 = \alpha_2 < 1$. In fact, for $\alpha_1 = \alpha_2 \to 0$ the correlation coefficient tends to an impulse at the origin. We note that the condition $\alpha_1 = \alpha_2 = 2$ corresponds to the Rayleigh case. Therefore, the correlation properties for the Weibull fading environment are roughly those of the Rayleigh ones in case the Weibull parameter is above 1. Conversely, the correlation properties differ substantially from the Rayleigh ones in case the Weibull parameter is below 1.

IX. Conclusion

Simple closed-form formulas for the joint statistics of two correlated Weibull variates are derived. These statistics are written in terms of well-known fading physical parameters. It has been found that the correlation properties for the Weibull fading environment are almost the same as those of the Rayleigh ones in case the Weibull parameter is above 1. On the other hand, they differ substantially from the Rayleigh ones in case the Weibull parameter is below 1.

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