# Integer Order Gaussian Moments

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In this work, integer order moments of Gaussian random variables are deduced.

## I. Gaussian Marginal Moment

Suppose a Gaussian random variable $X$, with mean $a_X$ and variance $\sigma_X^2$, and the zero mean variate $\tilde{X} = X - a_X$. Then, given $l \in \mathbb{N}$, it follows that

$$E\left\{X^l\right\} = \sum_{j=0}^{l} \left[ \binom{l}{j} a_X^{l-j} E\left\{\tilde{X}^j\right\} \right]. \tag{1}$$

The expected value of $\tilde{X}^j$, $j \in \mathbb{N}$, is given in [1, equação (5-73)]:

$$E\left\{\tilde{X}^j\right\} = \begin{cases} (j-1)!!\sigma_X^j & j \text{ even} \\ 0 & j \text{ odd} \end{cases}, \tag{2a}$$

where

$$(j-1)!! = \begin{cases} (j-1)\cdot(j-3)\cdot ... \cdot 6\cdot 4\cdot 2 & (j-1) \text{ even} \\ (j-1)\cdot(j-3)\cdot ... \cdot 5\cdot 3\cdot 1 & (j-1) \text{ odd} \end{cases}. \tag{2b}$$

From (1) and (2),

$$E\left\{X^l\right\} = a_X^l \sum_{j=0}^{\lfloor l/2 \rfloor} \left[ \binom{l}{2j} \frac{(2j-1)!!}{k_X^j} \right], \tag{3a}$$

where $k_X = a_X^2/\sigma_X^2$, and $\lfloor\cdot\rfloor$ is the largest integer less than or equal to the input value. Alternatively, using $(2j-1)!! = (2j)!/(2^j j!)$, $j \in \mathbb{N}$,

$$E\left\{X^l\right\} = a_X^l l! \sum_{j=0}^{\lfloor l/2 \rfloor} \left[ \frac{1}{(2k_X)^j (l-2j)! j!} \right]. \tag{3b}$$

Then, the ratio

$$\gamma_1(l; k_X) = \frac{E\left\{X^{2l}\right\}}{\sigma_X^{2l}} \tag{4a}$$

is

$$\gamma_1(l;k_X) = \frac{(2l)!}{2^l}\sum_{j=0}^{l}\left[\frac{(2k_X)^j}{(2j)!(l-j)!}\right]. \tag{4b}$$

## II. GAUSSIAN JOINT MOMENTS

### A. Two Random Variables

Suppose the joint Gaussian random variables $X$ and $Y$, respectively with means $a_X$ and $a_Y$, variances $\sigma_X^2$ and $\sigma_Y^2$, and correlation coefficient $\nu$, and also the zero mean variates $\tilde{X} = X - a_X$ and $\tilde{Y} - a_Y$. Then, given $l_i \in \mathbb{N}$, $i = 1, 2$, it follows that

$$E\left\{X^{2l_1}Y^{2l_2}\right\} = \sum_{j_1=0}^{2l_1}\sum_{j_2=0}^{2l_2}\left[\binom{2l_1}{j_1}\binom{2l_2}{j_2}a_X^{2l_1-j_1}a_Y^{2l_2-j_2}E\left\{\tilde{X}^{j_1}\tilde{Y}^{j_2}\right\}\right]. \tag{5}$$

Similarly, the ratios

$$\gamma_2(\mathbf{l};\mathbf{k};\nu) = \frac{E\left\{X^{2l_1}Y^{2l_2}\right\}}{(\sigma_X^{2l_1}\sigma_Y^{2l_2})} \tag{6a}$$

and

$$\varrho_2(\mathbf{j};\nu) = \frac{E\left\{\tilde{X}^{j_1}\tilde{Y}^{j_2}\right\}}{(\sigma_X^{j_1}\sigma_Y^{j_2})}, \tag{6b}$$

where $\mathbf{l} = [l_1 \;\; l_2] \in \mathbb{N}^2$, $\mathbf{j} = [j_1 \;\; j_2] \in \mathbb{N}^2$, and $\mathbf{k} = [k_X \;\; k_Y] = [a_X^2/\sigma_X^2 \;\; a_Y^2/\sigma_Y^2]$, are related by

$$\gamma_2(\mathbf{l};\mathbf{k};\nu) = k_X^{l_1}k_Y^{l_2}\sum_{j_1=0}^{2l_1}\sum_{j_2=0}^{2l_2}\left[\binom{2l_1}{j_1}\binom{2l_2}{j_2}\frac{\varrho_2(\mathbf{j};\nu)}{(k_X^{j_1}k_Y^{j_2})^{1/2}}\right]. \tag{6c}$$

The deduction of $\varrho_2(\mathbf{j};\nu)$ is performed departing from the JPDF of $\tilde{X}$ and $\tilde{Y}$,

$$f_{\tilde{X}\tilde{Y}}(\tilde{x},\tilde{y}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\nu^2}}\exp\left[-\frac{1}{2(1-\nu^2)}\left(\frac{\tilde{x}^2}{\sigma_X^2}+\frac{\tilde{y}^2}{\sigma_Y^2}-2\nu\frac{\tilde{x}\tilde{y}}{\sigma_X\sigma_Y}\right)\right], \tag{7a}$$

and from the definition of joint moment,

$$E\left\{\tilde{X}^{j_1}\tilde{Y}^{j_2}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{x}^{j_1}\tilde{y}^{j_2}f_{\tilde{X}\tilde{Y}}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}. \tag{7b}$$

Then, substituting (7a) into (7b), using $\varrho_2(\mathbf{j};\nu) = E\left\{\tilde{X}^{j_1}\tilde{Y}^{j_2}\right\}/(\sigma_X^{j_1}\sigma_Y^{j_2})$, and making some algebraic manipulations:

$$\varrho_2(\mathbf{j};\nu) = \frac{1}{\sqrt{2\pi}\sigma_X^{j_1}\sigma_Y^{j_2+1}} \int_{-\infty}^{\infty} \tilde{y}^{j_2} M_{\tilde{X}}(\tilde{y}) \exp\left(-\frac{\tilde{y}^2}{\sigma_Y^2}\right) d\tilde{y}, \tag{8a}$$

where

$$M_{\tilde{X}}(\tilde{y}) = \frac{1}{\sqrt{2\pi(1-\nu^2)}\sigma_X} \int_{-\infty}^{\infty} \tilde{x}^{j_1} \exp\left[-\frac{1}{2(1-\nu^2)\sigma_X^2}\left(\tilde{x} - \nu\frac{\sigma_X}{\sigma_Y}\tilde{y}\right)^2\right] d\tilde{x}. \tag{8b}$$

Observing that $M_{\tilde{X}}(\tilde{y})$ is the moment of order $j_1 \in \mathbb{N}$ of a Gaussian random variable with mean $\nu(\sigma_X/\sigma_Y)\tilde{y}$ and variance $\sqrt{(1-\nu^2)}\sigma_X$, and using (3b),

$$M_{\tilde{X}}(\tilde{y}) = j_1!\nu^{j_1}\sigma_X^{j_1} \sum_{n=0}^{\lfloor j_1/2 \rfloor} \left[\frac{\sigma_Y^{2n-j_1}\tilde{y}^{j_1-2n}}{2^n n!(j_1-2n)!}\left(\frac{1-\nu^2}{\nu^2}\right)^n\right]. \tag{9}$$

Replacing (9) into (8a), it follows that

$$\varrho_2(\mathbf{j};\nu) = j_1!\nu^{j_1} \sum_{n=0}^{\lfloor j_1/2 \rfloor} \left[\frac{\sigma_Y^{2n-j_1-j_2} M_{\tilde{Y}}}{2^n n!(j_1-2n)!}\left(\frac{1-\nu^2}{\nu^2}\right)^n\right], \tag{10a}$$

where

$$M_{\tilde{Y}} = \frac{1}{\sqrt{2\pi}\sigma_Y} \int_{-\infty}^{\infty} \tilde{y}^{j_1+j_2-2n} \exp\left(-\frac{\tilde{y}^2}{\sigma_Y^2}\right) d\tilde{y}. \tag{10b}$$

Observing that $M_{\tilde{Y}}$ is the moment of order $(j_1+j_2-2n) \in \mathbb{N}$ of a zero mean Gaussian random variable with variance $\sigma_Y$, and using (2),

$$M_{\tilde{Y}} = \begin{cases} (j_1+j_2-2n-1)!!\sigma_Y^{j_1+j_2-2n} & (j_1+j_2) \text{ even} \\ 0 & (j_1+j_2) \text{ odd} \end{cases}. \tag{11}$$

Thus, from (10a) and (11), it results that

$$\varrho_2(\mathbf{j};\nu) = \begin{cases} j_1!\nu^{j_1} \displaystyle\sum_{n=0}^{\lfloor j_1/2 \rfloor} \left[\frac{(j_1+j_2-2n-1)!!}{2^n n!(j_1-2n)!}\left(\frac{1-\nu^2}{\nu^2}\right)^n\right] & (j_1+j_2) \text{ even} \\ 0 & (j_1+j_2) \text{ odd} \end{cases}. \tag{12}$$

*B. Four Random Variables*

Suppose the joint Gaussian random variables $X_1$, $Y_1$, $X_2$, and $Y_2$, respectively with means $a_{X_1}$, $a_{Y_1}$, $a_{X_2}$, and $a_{Y_2}$, and variances $\sigma_{X_1}^2$, $\sigma_{Y_1}^2$, $\sigma_{X_2}^2$, and $\sigma_{Y_2}^2$. Besides, assume $X_i$ uncorrelated of $Y_i$, $i = 1, 2$, and the correlation coefficients

$$\nu_1 = \frac{Cov\{X_1, X_2\}}{\sigma_{X_1}\sigma_{X_2}} = \frac{Cov\{Y_1, Y_2\}}{\sigma_{Y_1}\sigma_{Y_2}} \tag{13a}$$

and

$$\nu_2 = \frac{Cov\{X_1, Y_2\}}{\sigma_{X_1}\sigma_{Y_2}} = -\frac{Cov\{Y_1, X_2\}}{\sigma_{Y_1}\sigma_{X_2}}. \tag{13b}$$

Now, consider $\tilde{X}_1 = X_1 - a_{X_1}$, $\tilde{Y}_1 = Y_1 - a_{Y_1}$, $\tilde{X}_2 = X_2 - a_{X_2}$, and $\tilde{Y}_2 = Y_2 - a_{Y_2}$. Then, given $l_i \in \mathbb{N}$, $i = 1, 2, 3, 4$, it follows that

$$\begin{aligned} E\left\{X_1^{2l_1}Y_1^{2l_2}X_2^{2l_3}Y_2^{2l_4}\right\} = &\sum_{j_1=0}^{2l_1}\sum_{j_2=0}^{2l_2}\sum_{j_3=0}^{2l_3}\sum_{j_4=0}^{2l_4}\left[\binom{2l_1}{j_1}\binom{2l_2}{j_2}\binom{2l_3}{j_3}\binom{2l_4}{j_4}\right. \\ &\left.\times a_{X_1}^{2l_1-j_1}a_{Y_1}^{2l_2-j_2}a_{X_3}^{2l_3-j_3}a_{Y_2}^{2l_4-j_4}E\left\{\tilde{X}_1^{j_1}\tilde{Y}_1^{j_2}\tilde{X}_2^{j_3}\tilde{Y}_2^{j_4}\right\}\right]. \end{aligned} \tag{14}$$

Analogously, the relation between the ratios

$$\gamma_4(\mathsf{l};\mathsf{k};\boldsymbol{\nu}) \triangleq \frac{E\left\{X_1^{2l_1}Y_1^{2l_2}X_2^{2l_3}Y_2^{2l_4}\right\}}{(\sigma_{X_1}^{2l_1}\sigma_{Y_1}^{2l_2}\sigma_{X_2}^{2l_3}\sigma_{Y_2}^{2l_4})} \tag{15a}$$

and

$$\varrho_4(\mathsf{j};\boldsymbol{\nu}) \triangleq \frac{E\left\{\tilde{X}_1^{j_1}\tilde{Y}_1^{j_2}\tilde{X}_2^{j_3}\tilde{Y}_2^{j_4}\right\}}{(\sigma_{X_1}^{j_1}\sigma_{Y_1}^{j_2}\sigma_{X_2}^{j_3}\sigma_{Y_2}^{j_4})}, \tag{15b}$$

where $\mathsf{l} = [l_1 \;\; l_2 \;\; l_3 \;\; l_4] \in \mathbb{N}^4$, $\mathsf{k} = [k_{X_1} \;\; k_{Y_1} \;\; k_{X_2} \;\; k_{Y_2}] = [a_{X_1}^2/\sigma_{X_1}^2 \;\; a_{Y_1}^2/\sigma_{Y_1}^2 \;\; a_{X_2}^2/\sigma_{X_2}^2 \;\; a_{Y_2}^2/\sigma_{Y_2}^2]$, $\boldsymbol{\nu} = [\nu_1 \;\; \nu_2]$, and $\mathsf{j} = [j_1 \;\; j_2 \;\; j_3 \;\; j_4] \in \mathbb{N}^4$, is

$$\gamma_4(\mathsf{l};\mathsf{k};\boldsymbol{\nu}) = k_{X_1}^{l_1}k_{Y_1}^{l_2}k_{X_2}^{l_3}k_{Y_2}^{l_4}\sum_{j_1=0}^{2l_1}\sum_{j_2=0}^{2l_2}\sum_{j_3=0}^{2l_3}\sum_{j_4=0}^{2l_4}\left[\binom{2l_1}{j_1}\binom{2l_2}{j_2}\binom{2l_3}{j_3}\binom{2l_4}{j_4}\frac{\varrho_4(\mathsf{j};\boldsymbol{\nu})}{(k_{X_1}^{j_1}k_{Y_1}^{j_2}k_{X_2}^{j_3}k_{Y_2}^{j_4})^{1/2}}\right]. \tag{16}$$

The deduction of $\varrho_4(\mathsf{j};\boldsymbol{\nu})$ is performed departing from the JPDF of $\tilde{\mathbf{X}} = [\tilde{X}_1 \;\; \tilde{X}_2]$ and $\tilde{\mathbf{Y}} = [\tilde{Y}_1 \;\; \tilde{Y}_2]$,

$$\begin{aligned} f_{\tilde{\mathbf{X}}\tilde{\mathbf{Y}}}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = &\frac{1}{4\pi^2\sigma_{X_1}\sigma_{Y_1}\sigma_{X_2}\sigma_{Y_2}(1-\rho^2)}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\tilde{x}_1^2}{\sigma_{X_1}^2}+\frac{\tilde{y}_1^2}{\sigma_{Y_1}^2}+\frac{\tilde{x}_2^2}{\sigma_{X_2}^2}+\frac{\tilde{y}_2^2}{\sigma_{Y_2}^2}\right.\right. \\ &\left.\left.-2\nu_1\frac{\tilde{x}_1\tilde{x}_2}{\sigma_{X_1}\sigma_{X_2}} - 2\nu_1\frac{\tilde{y}_1\tilde{y}_2}{\sigma_{Y_1}\sigma_{Y_2}} - 2\nu_2\frac{\tilde{x}_1\tilde{y}_2}{\sigma_{X_1}\sigma_{Y_2}} + 2\nu_2\frac{\tilde{y}_1\tilde{x}_2}{\sigma_{Y_1}\sigma_{X_2}}\right)\right], \end{aligned} \tag{17a}$$

where $\tilde{\mathbf{x}} = [\tilde{x}_1 \;\; \tilde{x}_2]$, $\tilde{\mathbf{y}} = [\tilde{y}_1 \;\; \tilde{y}_2]$, and $\rho = \sqrt{\nu_1^2 + \nu_2^2}$, and from the definition of joint moment,

$$E\left\{\tilde{X}_1^{j_1}\tilde{Y}_1^{j_2}\tilde{X}_2^{j_3}\tilde{Y}_2^{j_4}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{x}_1^{j_1}\tilde{y}_1^{j_2}\tilde{x}_2^{j_3}\tilde{y}_2^{j_4} f_{\tilde{\mathbf{X}}\tilde{\mathbf{Y}}}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})d\tilde{x}_1 d\tilde{y}_1 d\tilde{x}_2 d\tilde{y}_2. \tag{17b}$$

Then, substituting (17a) into (17b), using $\varrho_4(\mathsf{j};\boldsymbol{\nu}) = E\left\{\tilde{X}_1^{j_1}\tilde{Y}_1^{j_2}\tilde{X}_2^{j_3}\tilde{Y}_2^{j_4}\right\}/(\sigma_{X_1}^{j_1}\sigma_{Y_1}^{j_2}\sigma_{X_2}^{j_3}\sigma_{Y_2}^{j_4})$, and making some algebraic manipulations:

$$\varrho_4(\mathsf{j};\boldsymbol{\nu}) = \frac{(\sigma_{X_1}^{j_1}\sigma_{Y_1}^{j_2}\sigma_{X_2}^{j_3}\sigma_{Y_2}^{j_4})^{-1}}{2\pi\sigma_{X_2}\sigma_{Y_2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{x}_2^{j_3}\tilde{y}_2^{j_4} M_{\tilde{X}_1}(\tilde{x}_2,\tilde{y}_2) M_{\tilde{Y}_1}(\tilde{x}_2,\tilde{y}_2) \exp\left(-\frac{\tilde{x}_2^2}{\sigma_{X_2}^2} - \frac{\tilde{y}_2^2}{\sigma_{Y_2}^2}\right) d\tilde{x}_2 d\tilde{y}_2, \tag{18a}$$

where

$$M_{\tilde{X}_1}(\tilde{x}_2,\tilde{y}_2) = \frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_{X_1}} \int_{-\infty}^{\infty} \tilde{x}_1^{j_1} \exp\left[-\frac{1}{2(1-\rho^2)\sigma_{X_1}^2}\left(\tilde{x}_1 - \nu_1\frac{\sigma_{X_1}}{\sigma_{X_2}}\tilde{x}_2 - \nu_2\frac{\sigma_{X_1}}{\sigma_{Y_2}}\tilde{y}_2\right)^2\right] d\tilde{x}_1 \tag{18b}$$

$$M_{\tilde{Y}_1}(\tilde{x}_2,\tilde{y}_2) = \frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_{Y_1}} \int_{-\infty}^{\infty} \tilde{y}_1^{j_2} \exp\left[-\frac{1}{2(1-\rho^2)\sigma_{Y_1}^2}\left(\tilde{y}_1 - \nu_1\frac{\sigma_{Y_1}}{\sigma_{Y_2}}\tilde{y}_2 + \nu_2\frac{\sigma_{Y_1}}{\sigma_{X_2}}\tilde{x}_2\right)^2\right] d\tilde{y}_1. \tag{18c}$$

Note that $M_{\tilde{X}_1}(\tilde{x}_2,\tilde{y}_2)$ and $M_{\tilde{Y}_1}(\tilde{x}_2,\tilde{y}_2)$ are moments, respectively of orders $j_1 \in \mathbb{N}$ and $j_2 \in \mathbb{N}$, of Gaussian random variables, respectively with means $[\nu_1(\sigma_{X_1}/\sigma_{X_2})\tilde{x}_2 + \nu_2(\sigma_{X_1}/\sigma_{Y_2})\tilde{y}_2]$ and $[\nu_1(\sigma_{Y_1}/\sigma_{Y_2})\tilde{y}_2 - \nu_2(\sigma_{Y_1}/\sigma_{X_2})\tilde{x}_2]$ and variances $\sqrt{(1-\rho^2)}\sigma_{X_1}$ and $\sqrt{(1-\rho^2)}\sigma_{Y_1}$. Then, using (3b) for $M_{\tilde{X}_1}(\tilde{x}_2,\tilde{y}_2)$ and $M_{\tilde{Y}_1}(\tilde{x}_2,\tilde{y}_2)$,

$$M_{\tilde{X}_1}(\tilde{x}_2,\tilde{y}_2) = j_1!\sigma_{X_1}^{j_1} \sum_{n_1=0}^{\lfloor j_1/2 \rfloor} \left[\frac{(1-\rho^2)^{n_1}}{2^{n_1}n_1!(j_1-2n_1)!}\left(\nu_1\frac{\tilde{x}_2}{\sigma_{X_2}} + \nu_2\frac{\tilde{y}_2}{\sigma_{Y_2}}\right)^{j_1-2n_1}\right] \tag{19a}$$

$$M_{\tilde{Y}_1}(\tilde{x}_2,\tilde{y}_2) = j_2!\sigma_{Y_1}^{j_2} \sum_{n_2=0}^{\lfloor j_2/2 \rfloor} \left[\frac{(1-\rho^2)^{n_2}}{2^{n_2}n_2!(j_2-2n_2)!}\left(\nu_1\frac{\tilde{y}_2}{\sigma_{Y_2}} - \nu_2\frac{\tilde{x}_2}{\sigma_{X_2}}\right)^{j_2-2n_2}\right]. \tag{19b}$$

Applying the Newton's Binomial formula,

$$\left(\nu_1\frac{\tilde{x}_2}{\sigma_{X_2}} + \nu_2\frac{\tilde{y}_2}{\sigma_{Y_2}}\right)^{j_1-2n_1} = \sum_{n_3=0}^{j_1-2n_1}\left[\binom{j_1-2n_1}{n_3}\nu_1^{n_3}\nu_2^{j_1-2n_1-n_3}\frac{\tilde{x}_2^{n_3}\tilde{y}_2^{j_1-2n_1-n_3}}{\sigma_{X_2}^{n_3}\sigma_{Y_2}^{j_1-2n_1-n_3}}\right], \tag{20a}$$

$$\left(\nu_1\frac{\tilde{y}_2}{\sigma_{Y_2}} - \nu_2\frac{\tilde{x}_2}{\sigma_{X_2}}\right)^{j_2-2n_2} = \sum_{n_4=0}^{j_2-2n_2}\left[\binom{j_2-2n_2}{n_4}(-1)^{n_4}\nu_1^{j_2-2n_2-n_4}\nu_2^{n_4}\frac{\tilde{x}_2^{n_4}\tilde{y}_2^{j_2-2n_2-n_4}}{\sigma_{X_2}^{n_4}\sigma_{Y_2}^{j_2-2n_2-n_4}}\right]. \tag{20b}$$

From (18), (19), and (20),

$$\varrho_4(\mathrm{j};\boldsymbol{\nu}) = j_1!j_2!\nu_1^{j_2}\nu_2^{j_1}\sum_{n_1=0}^{\lfloor j_1/2\rfloor}\sum_{n_2=0}^{\lfloor j_2/2\rfloor}\sum_{n_3=0}^{j_1-2n_1}\sum_{n_4=0}^{j_2-2n_2}\left[\binom{j_1-2n_1}{n_3}\binom{j_2-2n_2}{n_4}\frac{1}{(j_1-2n_1)!(j_2-2n_2)!}\right.$$
$$\left.\times\frac{M_{\tilde{X}_2}M_{\tilde{Y}_2}}{\sigma_{X_2}^{j_3+n_3+n_4}\sigma_{Y_2}^{j_1+j_2+j_4-2n_1-2n_2-n_3-n_4}}\frac{(-1)^{n_4}(1-\rho^2)^{n_1+n_2}}{2^{n_1+n_2}n_1!n_2!\nu_1^{2n_2}\nu_2^{2n_1}}\left(\frac{\nu_1}{\nu_2}\right)^{n_3-n_4}\right], \tag{21a}$$

where

$$M_{\tilde{X}_2} = \frac{1}{\sqrt{2\pi}\sigma_{X_2}}\int_{-\infty}^{\infty}\tilde{x}_2^{j_3+n_3+n_4}\exp\left(-\frac{\tilde{x}_2^2}{\sigma_{X_2}^2}\right)d\tilde{x}_2 \tag{21b}$$
$$M_{\tilde{Y}_2} = \frac{1}{\sqrt{2\pi}\sigma_{Y_2}}\int_{-\infty}^{\infty}\tilde{y}_2^{j_1+j_2+j_4-2n_1-2n_2-n_3-n_4}\exp\left(-\frac{\tilde{y}_2^2}{\sigma_{Y_2}^2}\right)d\tilde{y}_2. \tag{21c}$$

Note that $M_{\tilde{X}_2}$ and $M_{\tilde{Y}_2}$ are moments, respectively of orders $(j_3+n_3+n_4)\in\mathbb{N}$ and $(j_1+j_2+j_4-2n_1-2n_2-n_3-n_4)\in\mathbb{N}$, of zero mean Gaussian random variables, respectively with variances $\sigma_{X_2}$ and $\sigma_{Y_2}$. Therefore, using (2) for $M_{\tilde{X}_2}$ and $M_{\tilde{Y}_2}$

$$M_{\tilde{X}_2} = \begin{cases}(j_3+n_3+n_4-1)!!\sigma_{X_2}^{j_3+n_3+n_4} & (j_3+n_3+n_4)\text{ even}\\ 0 & (j_3+n_3+n_4)\text{ odd}\end{cases} \tag{22a}$$
$$M_{\tilde{Y}_2} = \begin{cases}\dfrac{(j_1+j_2+j_4-2n_1-2n_2-n_3-n_4-1)!!}{\sigma_{Y_2}^{-(j_1+j_2+j_4-2n_1-2n_2-n_3-n_4)}} & (j_1+j_2+j_4-n_3-n_4)\text{ even}\\ 0 & (j_1+j_2+j_4-n_3-n_4)\text{ odd}\end{cases}. \tag{22b}$$

Thus, the product $M_{\tilde{X}_2}M_{\tilde{Y}_2}$ is

$$M_{\tilde{X}_2}M_{\tilde{Y}_2} = \frac{(j_3+n_3+n_4-1)!!(j_1+j_2+j_4-2n_1-2n_2-n_3-n_4-1)!!}{\sigma_{X_2}^{-(j_3+n_3+n_4)}\sigma_{Y_2}^{-(j_1+j_2+j_4-2n_1-2n_2-n_3-n_4)}} \tag{23a}$$

if both $(j_1+j_2+j_3+j_4)$ and $(j_3+n_3+n_4)$ are even, and

$$M_{\tilde{X}_2}M_{\tilde{Y}_2} = 0 \tag{23b}$$

otherwise.

From (21) and (23),

$$\varrho_4(\mathrm{j};\boldsymbol{\nu}) = j_1!j_2!\nu_1^{j_2}\nu_2^{j_1}\sum_{n_1=0}^{\lfloor j_1/2\rfloor}\sum_{n_2=0}^{\lfloor j_2/2\rfloor}\sum_{n_3=0}^{j_1-2n_1}\sum_{n_4=0}^{j_2-2n_2}\left[\frac{(j_1+j_2+j_4-2n_1-2n_2-n_3-n_4-1)!!}{(j_1-2n_1-n_3)!(j_2-2n_2-n_4)!}\right.$$
$$\left.\times\frac{[(-1)^{j_3+n_3}+(-1)^{n_4}](j_3+n_3+n_4-1)!!}{2^{n_1+n_2+1}n_1!n_2!n_3!n_4!}\frac{(1-\rho^2)^{n_1+n_2}}{\nu_1^{2n_2}\nu_2^{2n_1}}\left(\frac{\nu_1}{\nu_2}\right)^{n_3-n_4}\right] \tag{24a}$$

for $(j_1 + j_2 + j_3 + j_4)$ even, and

$$\varrho_4(\mathsf{j}; \boldsymbol{\nu}) = 0 \tag{24b}$$

for $(j_1 + j_2 + j_3 + j_4)$ odd.

## References

[1] A. Papoulis and S. U. Pillai, *Probability, Random Varialbes and Stochastic Process*, 4th ed. McGraw-Hill, 2002.