Second-Order Statistics of Maximal-Ratio and Equal-Gain Combining in Weibull Fading

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***Abstract*— Exact expressions for the level crossing rate and average fade duration of M-branch equal-gain and maximal-ratio combining systems in a Weibull fading environment are presented. The expressions apply to unbalanced, non-identical, independent diversity channels. In addition, new closed-form solutions for some special cases are obtained.**

***Index Terms*— Average fade duration, equal-gain combining, level crossing rate, maximal-ratio combining, and Weibull fading channels.**

I. Introduction

LEVEL crossing rate (LCR) and average fade duration (AFD) are important second-order statistical quantities, which have been extensively explored in the literature. Closed-form expressions of these statistics for a single-channel for the well-known fading environments, such as Rayleigh, Rice, Hoyt, Nakagami-m and Weibull can be found in [1], [2], [3], [4], [5].

This paper derives *exact* LCR and AFD expressions for the Weibull channel in diversity systems using equal-gain combining (EGC) and maximal-ratio combining (MRC). The formulas apply to M unbalanced, non-identical, independent branches and have been validated by specializing the general results to some particular cases whose solutions are known and, more generally, by means of simulation. In addition, new closed-form solutions for some special cases are also presented.

II. Preliminaries

A. The Weibull Physical Model

The Weibull distribution is an empirical distribution, which was first proposed aiming at applications in reliability engineering. It has also found use in wireless communications to model the fading envelope. Due to the lack of a theoretical basis [6], the application of the Weibull distribution in wireless communications has been limited to the first order statistics of the fading signal. In [7], a very simple physical model for the Weibull distribution was proposed. In essence, in the proposed model the Weibull envelope R_i at the ith branch, $i = 1, \ldots, M$, is obtained as a non-linear function of the modulus of multipath components, the non-linearity expressed in terms of a parameter $\alpha_i > 0$, i.e.,

$$R_i = \left(X_i^2 + Y_i^2\right)^{1/\alpha_i} \tag{1}$$

where X_i and Y_i are independent zero-mean Gaussian variates with identical variances σ^2. The probability density function (PDF) $p_{R_i}(\cdot)$ of R_i is given by

$$p_{R_i}(r_i) = \frac{\alpha_i r_i^{\alpha_i - 1}}{\Omega_i} \exp\left(-\frac{r_i^{\alpha_i}}{\Omega_i}\right) \tag{2}$$

where $\Omega_i = \mathrm{E}[R_i^{\alpha_i}]$. For isotropic scattering, the time derivatives $\dot{X}_i$ and $\dot{Y}_i$ of X_i and Y_i, respectively, are known to be zero-mean Gaussian variates with variances $\left(\sqrt{2}\pi f_m\right)^2 \sigma^2$ [1]. Correspondingly, the conditional PDF (CPDF) $p_{\dot{R}_i|R_i}(\cdot|\cdot)$ of $\dot{R}_i$ (the time derivative of R_i) given R_i is easily found from (1) as

$$p_{\dot{R}_i|R_i}(\dot{r}_i|r_i) = \frac{1}{\sqrt{2\pi}\sigma_{\dot{R}_i}} \exp\left(-\frac{1}{2}\left(\frac{\dot{r}_i}{\sigma_{\dot{R}_i}}\right)^2\right) \tag{3}$$

where $\sigma_{\dot{R}_i}^2 = \left(\frac{2\pi f_m}{\alpha_i}\right)^2 \Omega_i r_i^{2-\alpha_i}$ and f_m is the maximum Doppler shift.

B. LCR and AFD

The LCR $n_R(r)$ and AFD $T_R(r)$ of a signal R at level r are respectively given by

$$n_R(r) = \int_0^\infty \dot{r} p_{R,\dot{R}}(r, \dot{r}) d\dot{r} \quad \text{and} \quad T_R(r) = \frac{P_R(r)}{n_R(r)} \tag{4}$$

where $p_{R,\dot{R}}(\cdot,\cdot)$ is the joint PDF (JPDF) of R and its time derivative $\dot{R}$, and $P_R(\cdot)$ is the cumulative distribution function (CDF) of R. In the following, R shall be taken as the combiner output and (4) shall be used to derive the LCR and AFD for M-branch EGC and MRC in a Weibull fading environment.

III. Equal-Gain Combining

In EGC, the received signals are cophased and added so that R and $\dot{R}$, already taking into account the resultant output noise power, are written as

$$R = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} R_i \text{ and } \dot{R} = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} \dot{R}_i \tag{5}$$

The CDF of R can be calculated by integrating the JPDF of R_i, $i = 1, \ldots, M$, over the M-dimensional volume bounded by the hyperplane $\sqrt{M}r = \sum_{i=1}^{M} r_i$ and the coordinate hyperplanes. Using a procedure found in [4]

$$P_R(r) = \int_0^{\sqrt{M}r} \int_0^{\sqrt{M}r - r_M} \cdots \int_0^{\sqrt{M}r - \sum_{i=3}^{M} r_i} \int_0^{\sqrt{M}r - \sum_{i=2}^{M} r_i} \times \prod_{i=1}^{M} p_{R_i}(r_i) dr_1 dr_2 \cdots dr_{M-1} dr_M \tag{6}$$

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$$p_{R,\dot{R}}(r,\dot{r}) = \sqrt{M}\overbrace{\int_0^{\sqrt{M}r}\int_0^{\sqrt{M}r-r_M}\cdots\int_0^{\sqrt{M}r-\sum_{i=3}^{M}r_i}}^{M-1} \times p_{R_1,R_2,\ldots,R_M,\dot{R}}\left(\left(\sqrt{M}r-\sum_{i=2}^{M}r_i\right),r_2,\ldots,r_M,\dot{r}\right)dr_2\cdots dr_{M-1}dr_M \quad (7)$$

where $p_{R_i}(\cdot)$ is given by (2). Note from (3) and (5) that $p_{\dot{R}|R_1,\ldots,R_M}(\cdot|\cdot,...,\cdot)$, the CPDF of $\dot{R}$ given R_i, $i=1,\ldots,M$, is zero-mean Gaussian distributed with variance $\sigma^2_{\dot{R}} = \sum_{i=1}^{M}\sigma^2_{\dot{R}_i}/M$. Derivating (6) with respect to r to obtain $p_R(r)$ as in [4] and then using the Bayes' rule, $p_{R,\dot{R}}(\cdot,\cdot)$ can be found as (7), where $p_{R_1,\ldots,R_M,\dot{R}}(\cdot,\ldots,\cdot,\cdot)$ is the JPDF of R_i, $i=1,\ldots,M$, and $\dot{R}$. Of course,

$$p_{R_1,\ldots,R_M,\dot{R}}(r_1,...,r_M,\dot{r}) = p_{\dot{R}|R_1,\ldots,R_M}(\dot{r}|r_1,...,r_M) \times p_{R_1,\ldots,R_M}(r_1,...,r_M) \quad (8)$$

where $p_{\dot{R}|R_1,\ldots,R_M}(\cdot|\cdot,...,\cdot)$ is as already mentioned and $p_{R_1,\ldots,R_M}(r_1,...,r_M) = \prod_{i=1}^{M}p_{R_i}(r_i)$, since the branches are independent. Using (8) into (7) and (4) appropriately, the output LCR of an M-branch EGC system in a Weibull fading environment can be finally written as (9), where $p_{R_i}(\cdot)$ is given by (2). From (4), (6), and (9), the output AFD of EGC in multi-branch Weibull fading is then obtained.

IV. Maximal-Ratio Combining

In MRC, the received signals are cophased, each signal is amplified appropriately for optimal combining, and the resultant signals are added. The combiner output envelope R and its time derivative $\dot{R}$ are written as

$$R^2 = \sum_{i=1}^{M}R_i^2 \text{ and } \dot{R} = \sum_{i=1}^{M}\frac{R_i}{R}\dot{R}_i \quad (10)$$

The MRC analysis follows exactly the same steps detailed for EGC in the previous section. The hyperplane used to compute $P_R(\cdot)$, however, is $r^2=\sum_{i=1}^{M}r_i^2$. In addition $\sigma^2_{\dot{R}} = \sum_{i=1}^{M}R_i^2\sigma^2_{\dot{R}_i}/R^2$. The resulting $P_R(\cdot)$, $p_{R,\dot{R}}(\cdot,\cdot)$, and $n_R(\cdot)$ are given by (11), (12), and (13), respectively.

$$P_R(r) = \int_0^r\int_0^{\sqrt{r^2-r_M^2}}\cdots\int_0^{\sqrt{r^2-\sum_{i=3}^{M}r_i^2}}\int_0^{\sqrt{r^2-\sum_{i=2}^{M}r_i^2}} \times \prod_{i=1}^{M}p_{R_i}(r_i)\,dr_1dr_2\ldots dr_{M-1}dr_M \quad (11)$$

From (4), (11), and (13), the output AFD of MRC in a multi-branch Weibull fading is then obtained.

V. Special Cases

The formulation developed here can be specialized to yield closed-form expressions. In addition they can also be particularized in order to match those already found in the literature.

A. New Closed-Form Expressions

The closed-form expressions shown in this subsection have been obtained for balanced branches, i.e. $\Omega_i=\Omega$, and same fading parameter, i.e. $\alpha_i=\alpha$, $i=1,\ldots,M$. Define $\rho = r/\sqrt[\alpha]{\Omega}$.

- For EGC and $\alpha=1$

$$n_R(r) = \frac{\sqrt{2\pi}f_m M^{\frac{2M-1}{4}}\rho^{M-\frac{1}{2}}}{\Gamma(M)\exp\left(\sqrt{M}\rho\right)} \quad (14)$$

$$T_R(r) = \frac{\Gamma(M,\sqrt{M}\rho)\exp\left(\sqrt{M}\rho\right)}{\sqrt{2\pi}f_m M^{\frac{2M-1}{4}}\rho^{M-\frac{1}{2}}} \quad (15)$$

where $\Gamma(a,z)=\int_0^z t^{a-1}e^{-t}dt$ is the incomplete gamma function.

- For MRC, dual branch ($M=2$) and $\alpha=4$

$$n_R(r) = e^{-\rho^4}\sqrt{\pi}\left(\rho^2+\sqrt{\frac{\pi}{2}}\left(\rho^4-1\right)e^{\frac{\rho^4}{2}}\mathrm{erf}\left(\frac{\rho^2}{\sqrt{2}}\right)\right) \quad (16)$$

$$T_R(r) = \frac{\sqrt{2}\left(e^{\rho^4}-1\right)-\sqrt{\pi}\rho^2e^{\frac{\rho^4}{2}}\mathrm{erf}\left(\frac{\rho^2}{\sqrt{2}}\right)}{\sqrt{2\pi}\rho^2+\pi\left(\rho^4-1\right)e^{\frac{\rho^4}{2}}\mathrm{erf}\left(\frac{\rho^2}{\sqrt{2}}\right)} \quad (17)$$

where erf $(\cdot)$ is the error function. The expressions for the unbalanced diversity channels have also been obtained, but they are too long to be presented here.

B. Results from the Literature

For $M=1$, the results coincide with those of [5, Eqs. 12 and 13]. For balanced diversity channels and $\alpha_i=2$, $i=1,...,M$, the formulations reduce to the M-branch EGC and MRC of the identically, independently distributed Rayleigh case, given by [4, Eqs. 23 and 24] for $m=1$ and [4, Eqs. 38 and 39], respectively.

VI. Some Plots

For the more general cases, including identical and non-identical fading branches, *exhaustive* simulations have been carried out and compared with the analytical expressions obtained here. All the cases investigated revealed an excellent agreement between analytical and simulation results. Figs. 1 and 2 show the LCR and the AFD of EGC and MRC, respectively, for $M=1,2,4$ and $\alpha_i=2,3,4$, considering identical Weibull-fading channels. For the sake of clarity, the simulation data have been omitted in the figures. In fact, they are practically coincident with the theoretical curves.

$$n_R(r)=\sqrt{2\pi}f_m\overbrace{\int_0^{\sqrt{M}r}\int_0^{\sqrt{M}r-r_M}\cdots\int_0^{\sqrt{M}r-\sum_{i=3}^{M}r_i}}^{M-1}\sqrt{\frac{\left(\sqrt{M}r-\sum_{i=2}^{M}r_i\right)^{2-\alpha_1}\Omega_1}{\alpha_1^2}+\sum_{i=2}^{M}\frac{r_i^{2-\alpha_i}\Omega_i}{\alpha_i^2}}$$
$$\times\, p_{R_1}\left(\sqrt{M}r-\sum_{i=2}^{M}r_i\right)\prod_{i=2}^{M}p_{R_i}(r_i)dr_2\cdots dr_{M-1}dr_M \quad (9)$$

$$p_{R,\dot{R}}(r,\dot{r})=\overbrace{\int_0^{r}\int_0^{\sqrt{r^2-r_M^2}}\cdots\int_0^{\sqrt{r^2-\sum_{i=3}^{M}r_i^2}}}^{M-1}\frac{r}{\sqrt{r^2-\sum_{i=2}^{M}r_i^2}}$$
$$\times\, p_{R_1,R_2,\ldots,R_M,\dot{R}}\left(\left(\sqrt{r^2-\sum_{i=2}^{M}r_i^2}\right),r_2,\ldots,r_M,\dot{r}\right)dr_2\cdots dr_{M-1}dr_M \quad (12)$$

$$n_R(r)=\sqrt{2\pi}f_m\overbrace{\int_0^{r}\int_0^{\sqrt{r^2-r_M^2}}\cdots\int_0^{\sqrt{r^2-\sum_{i=3}^{M}r_i^2}}}^{M-1}\frac{1}{\sqrt{r^2-\sum_{i=2}^{M}r_i^2}}\sqrt{\frac{\left(r^2-\sum_{i=2}^{M}r_i^2\right)^{\frac{4-\alpha_1}{2}}\Omega_1}{\alpha_1^2}+\sum_{i=2}^{M}\frac{r_i^{4-\alpha_i}\Omega_i}{\alpha_i^2}}$$
$$\times\, p_{R_1}\left(\sqrt{r^2-\sum_{i=2}^{M}r_i^2}\right)\prod_{i=2}^{M}p_{R_i}(r_i)dr_2\cdots dr_{M-1}dr_M \quad (13)$$

VII. Conclusions

Exact formulas for level crossing rate and average fade duration for M-branch EGC, MRC techniques in a Weibull environment have been obtained. The formulas have been validated by reducing the general case to some special cases for which the solutions are known and, more generally, by means of simulation. Closed-form expressions for some particular cases have also been presented.

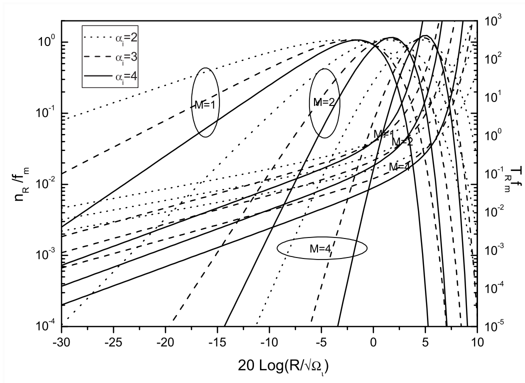

Fig. 1. LCR and AFD of EGC for identical Weibull-fading channels ($M=1$, 2, 4 and $\alpha_i=2$, 3, 4)

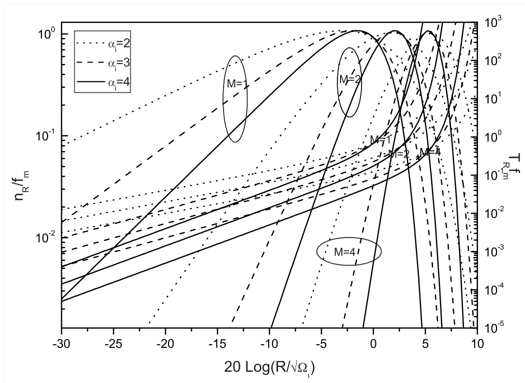

Fig. 2. LCR and AFD of MRC for identical Weibull-fading channels ($M=1$, 2, 4 and $\alpha_i=2$, 3, 4)

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