Generalized Nakagami-*m* Phase Crossing Rate

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***Abstract*— This paper provides *simple, exact, new* closed-form expressions for the generalized phase crossing rate of Nakagami-*m* fading channels. Sample numerical results obtained by simulation are presented that validate the formulations developed here. A special case of this formulation is the Rayleigh case, whose result agrees with that obtained elsewhere in the literature. In passing, several *new* closed-form results concerning the statistics of the envelope, its in-phase and quadrature components, phase, and their time derivatives are obtained.**

***Index Terms*— Phase crossing rate, Nakagami-*m* fading channels.**

I. Introduction

In wireless communications, envelope and phase of a received signal vary in a random manner because of multipath fading. The behavior of the envelope in a fading channel has been extensively explored in the literature. On the other hand, although the knowledge of the phase variation of the received signal plays a crucial role in the design of any communication technique, its characterization for an important fading channel, namely Nakagami-*m*, remains unknown. The study of the phase behavior may be useful, for instance, in the design of optimal carrier recovery schemes needed in the synchronization subsystem of coherent receivers [1]. A pioneering work in this matter was carried out by Rice in his classical paper [2], in which the aim was to evaluate the click noise in FM systems, assuming the noise spectrum to be symmetric about the sine wave frequency. In this sense, Rice obtained the phase crossing rate at the particular phase levels $\theta = 0$ and $\theta = \pi$ for the envelope lying within an arbitrary range. In [3], the work by Rice was extended to consider asymmetrical noise spectrum as well as arbitrary phase levels, i.e., $-\pi \leq \theta \leq \pi$. More recently, [4] and [5] investigated the phase crossing statistics, respectively, for the Hoyt (Nakagami-q) and Weibull processes. In this work, we define PCR as the usual phase crossing rate and GPCR as the phase crossing rate conditioned on the envelope lying within an arbitrary range.

This Letter provides *simple, exact, new* closed-form expressions for the GPCR of Nakagami-*m* fading channels. Clearly, the PCR is also attained as a special case. Exhaustive simulations fully validate the formulation proposed here. Our formulation makes use of the fading model recently presented in [6], in which a phase-envelope joint probability distribution for the Nakagami-*m* fading has been proposed. In passing, several *new* closed-form results concerning the statistics of the envelope, its in-phase and quadrature components, phase, and their time derivatives are obtained.

This paper is organized as follows. Section II revisits the phase-envelope joint fading model of [6]. Section III derives the joint statistics of the envelope, phase, and their time derivatives. Section IV derives the GPCR. Section V presents sample numerical results. Finally, Section VI draws some conclusions.

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II. The Nakagami-*m* Phase-Envelope Joint Model Revisited

Let R and Θ be random variables representing, respectively, the envelope and phase of the Nakagami-*m* signal. The Nakagami-*m* phase-envelope joint probability density function (JPDF) $f_{R,\Theta}(r,\theta)$ is given by [6]

$$f_{R,\Theta}(r,\theta) = \frac{m^m |\sin(2\theta)|^{m-1}\, r^{2m-1}}{2^{m-1}\Omega^m\, \Gamma^2(\frac{m}{2})} \exp\left(-\frac{mr^2}{\Omega}\right) \quad (1)$$

where $\Omega = \mathrm{E}[R^2]$, $m = \mathrm{E}^2[R^2]/(\mathrm{E}[R^4] - \mathrm{E}^2[R^2])$, $\Gamma(\cdot)$ is the Gamma function [7, Eq. 6.1.1], and $\mathrm{E}[\cdot]$ denotes the expectation operator. According to this model [6], assuming X and Y as, respectively, the independent in-phase and quadrature components of the Nakagami-*m* signal, their PDF $f_Z(z)$, $Z = X$ or $Z = Y$, is given by

$$f_Z(z) = \frac{m^{\frac{m}{2}}\, |z|^{m-1}}{\Omega^{\frac{m}{2}}\, \Gamma(\frac{m}{2})} \exp\left(-\frac{mz^2}{\Omega}\right), -\infty < z < \infty \quad (2)$$

From [6], $Z = S\,|Z|$, where S stands for $sgn\,(Z)$ (sign of Z) and $|Z|$ is Nakagami-*m* distributed. Then, for mathematical simplicity, we write $Z = SN$, where N denotes a Nakagami-*m* variate.

III. Joint Statistics of the Envelope, Phase, and their Derivatives

Let $\dot{Z}$ be the time derivative of Z. From the above, $\dot{Z} = \dot{S}N + S\dot{N}$. Because S assumes the constant values ± 1, except for the transition instants ($-1 \rightarrow +1$ and $+1 \rightarrow -1$), its time derivative $\dot{S}$ is nil. In addition, because Z is continuous, the transition instants occur exactly and only at the zero crossing instants of Z, when $N = |Z|$ is nil. Therefore, $\dot{S}N = 0$ always and $\dot{Z} = S\dot{N}$. It has been shown in [8] that $\dot{N}$ is independent of N and Gaussian distributed with zero mean and standard deviation $\dot{\sigma} = \pi f_d \sqrt{\Omega/m}$ (f_d is the maximum Doppler shift in Hz). Knowing that $\dot{Z} = S\dot{N}$, $\dot{Z}$ is also Gaussian distributed conditioned on $Z = SN$, having the same distribution parameters as $\dot{N}$. Consequently, $\dot{Z}$ is independent of Z. More specifically, X is independent of $\dot{X}$, and Y is independent of $\dot{Y}$. From the proposed model [6], X and Y are independent processes. Thus, X is independent of $\dot{Y}$ and Y is independent of $\dot{X}$. Using (2) for Z and knowing that $\dot{Z}$ is Gaussian distributed with the cited parameters, then the JPDF $f_{X,\dot{X},Y,\dot{Y}}(x,\dot{x},y,\dot{y})$ is given by

$$f_{X,\dot{X},Y,\dot{Y}}(x,\dot{x},y,\dot{y}) = f_X(x) f_{\dot{X}}(\dot{x}) f_Y(y) f_{\dot{Y}}(\dot{y}) = \frac{m^{m+1}\,|x|^{m-1}|y|^{m-1}}{\Omega^{m+1}\,\Gamma^2(\frac{m}{2})\,2\pi^3\,f_d^2} \times \exp\left(-\frac{m}{\Omega}\left(x^2+y^2+\frac{1}{2\pi^2 f_d^2}\dot{x}^2+\frac{1}{2\pi^2 f_d^2}\dot{y}^2\right)\right) \tag{3}$$

From [6], $X = R\cos\Theta$ and $Y = R\sin\Theta$. Therefore, $\dot{X} = \dot{R}\cos\Theta - R\dot{\Theta}\sin\Theta$ and $\dot{Y} = \dot{R}\sin\Theta + R\dot{\Theta}\cos\Theta$. Following the standard statistical procedure for the transformation of variates and after algebraic manipulations, the JPDF $f_{R,\dot{R},\Theta,\dot{\Theta}}(r,\dot{r},\theta,\dot{\theta})$ of the envelope, the phase, and their respective time derivatives, is obtained as

$$f_{R,\dot{R},\Theta,\dot{\Theta}}(r,\dot{r},\theta,\dot{\theta}) = \frac{m^{m+1}\,r^{2m}\,|\sin(2\theta)|^{m-1}}{\Omega^{m+1}\,\Gamma^2(\frac{m}{2})\,2^m\pi^3 f_d^2} \times \exp\left(-\frac{m}{\Omega}\left(r^2+\frac{\dot{r}^2+r^2\,\dot{\theta}^2}{2\pi^2\,f_d^2}\right)\right) \tag{4}$$

For $m = 1$, (4) reduces to the Rayleigh fading case given in [9, Eq. 1.3-33]. Note also that $f_{R,\dot{R},\Theta,\dot{\Theta}}(r,\dot{r},\theta,\dot{\theta}) = f_{\dot{R}}(\dot{r}) f_\Theta(\theta) f_{R,\dot{\Theta}}(r,\dot{\theta})$. Some important densities are obtained by performing the appropriate integration in (4) and they are shown as follows:

$$f_{R,\Theta,\dot{\Theta}}(r,\theta,\dot{\theta}) = \frac{m^{m+\frac{1}{2}}\,r^{2m}|\sin(2\theta)|^{m-1}}{2^{m-\frac{1}{2}}\Omega^{m+\frac{1}{2}}\,\Gamma^2(\frac{m}{2})\,f_d\,\pi^{3/2}} \times \exp\left(-\frac{mr^2}{\Omega}\left(1+\frac{\dot{\theta}^2}{2\pi^2\,f_d^2}\right)\right) \tag{5}$$

$$f_{\Theta,\dot{\Theta}}(\theta,\dot{\theta}) = \frac{|\sin(2\theta)|^{m-1}\,\Gamma(m+\frac{1}{2})}{2^{m+\frac{1}{2}}\left(1+\frac{\dot{\theta}^2}{2\pi^2\,f_d^2}\right)^{m+\frac{1}{2}}\Gamma^2(\frac{m}{2})\,f_d\,\pi^{3/2}} \tag{6}$$

$$f_{\dot{\Theta}}(\dot{\theta}) = \frac{\Gamma(m+\frac{1}{2})}{\sqrt{2}\left(1+\frac{\dot{\theta}^2}{2\pi^2\,f_d^2}\right)^{m+\frac{1}{2}}\Gamma(m)\,f_d\,\pi^{3/2}} \tag{7}$$

The distribution $F_{\dot{\Theta}}(\dot{\theta})$ of $\dot{\Theta}$ is obtained as

$$F_{\dot{\Theta}}(\dot{\theta}) = \frac{1}{2} + \frac{\dot{\theta}\,\Gamma(m+\frac{1}{2})\,{}_2F_1\left(m,\frac{1}{2}+m;\frac{3}{2};-\frac{\dot{\theta}^2}{2\pi^2 f_d^2}\right)}{\sqrt{2}\,f_d\,\pi^{3/2}\Gamma(m)} \tag{8}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function [7, Eq. 15.1.1]. Next, we derive the GPCR of Nakagami-*m* fading channels.

IV. Generalized PCR

The PCR, denoted by $N_\Theta(\theta)$, is defined as the average number of upward (or downward) crossings per second at a specific phase level θ. This definition can be extended to a general case (GPCR), in which the crossing rate of the phase is conditioned on $r_1 \le R \le r_2$, as performed by Rice [2] for the click noise problem investigation in FM systems. Thus, the GPCR of Nakagami-*m* channels can be formulated as

$$N_{\Theta|R}(\theta;r_1,r_2) = \int_0^\infty \dot{\theta} f_{\Theta,\dot{\Theta};R}(\theta,\dot{\theta}|r_1 \le R \le r_2)d\dot{\theta} = \frac{\int_{r_1}^{r_2}\int_0^\infty \dot{\theta} f_{R,\Theta,\dot{\Theta}}(r,\theta,\dot{\theta})d\dot{\theta}dr}{F_R(r_2)-F_R(r_1)} \tag{9}$$

In (9), $F_R(\cdot)$ is the distribution of R, which, for the Nakagami-*m* case, is given by $F_R(r) = \gamma(m, mr^2/\Omega)/\Gamma(m)$, where $\gamma(\cdot,\cdot)$ is the incomplete Gamma function [7, Eq. 6.5.2]. By the appropriate substitutions and carrying out the necessary algebraic manipulations

$$N_{\Theta|R}(\theta;r_1,r_2) = \frac{\sqrt{\pi}f_d\,|\sin(2\theta)|^{m-1}}{2^{m+\frac{1}{2}}\,\Gamma^2(\frac{m}{2})} \times \frac{\Gamma(-\frac{1}{2}+m;m\rho_1^2,m\rho_2^2)\Gamma(m)}{\Gamma(m,m\rho_2^2,m\rho_1^2)} \tag{10}$$

where $\Gamma(a;b,c) = \gamma(a,b)-\gamma(a,c)$ is the generalized incomplete Gamma function and $\rho_i^2 = r_i^2/\Omega$, $i = 1,2$. In particular, for $m = 1$, we obtain the generalized PCR for Rayleigh fading channels, which can be expressed as

$$N_{\Theta|R}(\theta;r_1,r_2) = \frac{f_d\,\Gamma(\frac{1}{2};\rho_1^2,\rho_2^2)}{2\sqrt{2\pi}\,(\exp(-\rho_1^2)-\exp(-\rho_2^2))} \tag{11}$$

For the specific case in which $r_1 = 0$ and $r_2 = \infty$

$$N_\Theta(\theta) = \frac{\sqrt{\pi}\,f_d\,|\sin(2\theta)|^{m-1}\,\Gamma(m-\frac{1}{2})}{2^{m+\frac{1}{2}}\Gamma^2(\frac{m}{2})} \tag{12}$$

For $m = 1$ (Rayleigh case), (12) yields

$$N_\Theta(\theta) = \frac{f_d}{2\sqrt{2}} \tag{13}$$

Note that (13) is independent of the phase level θ, which is coherent with the result obtained by Rice [2].

V. Numerical Results

In this section, some plots illustrate the formulations obtained. In addition, the validity of the proposed formulations is checked by comparing the theoretical curves against the simulation results. As will be observed, an *excellent* agreement has been achieved between the simulation results and the formulation proposed here. In Figs. 1 and 2, the normalized PCR is depicted for several fading conditions. For $m = 1$ (Rayleigh), this statistics is independent of the phase level, assuming a constant value equal to $1/(2\sqrt{2})$, which is coherent with the result obtained by Rice [2]. For $m = 1.5, 2, 2.5, 4, 4.5$ the normalized PCR is periodic with period $\pi/2$ and nil for integers multiples of $\pi/2$. Note the excellent agreement between the theoretical and simulated curves.

For $m = 2$, Fig. 3 shows the normalized GPCR. For $\rho_1 = 0$ and varying ρ_2, we note that the curves do not differ substantially one from another for $\rho_2 = 2$ and $\rho_2 = 50$. In fact, it has been observed that the curve for $\rho_2 = 2$ is practically coincident with that for which $\rho_2 \to \infty$.

Fig. 4 depicts the PCR for several fading parameters. For values of m higher than 1, the maximum of the curves is reached at odd multiples of $\pi/4$. For values of m between 0.5 and 1, the curves are convex with minima at odd multiples

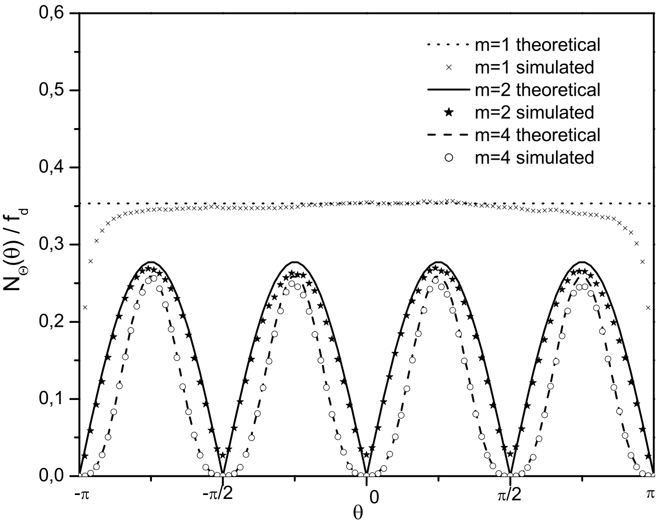

Fig. 1. Comparison between simulated and theoretical curves for the normalized PCR.

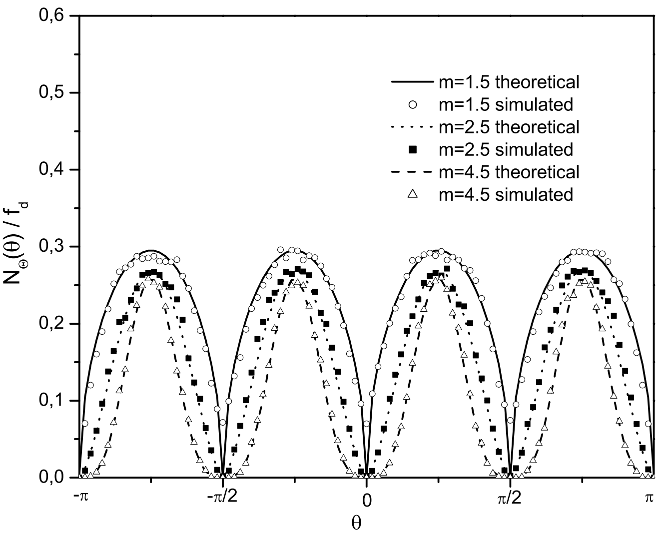

Fig. 2. Comparison between simulated and theoretical curves for the normalized PCR.

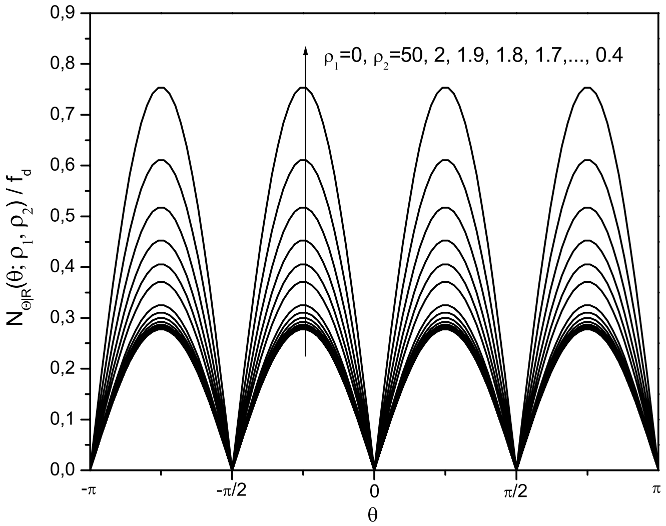

Fig. 3. Normalized GPCR for a fading parameter = 2.

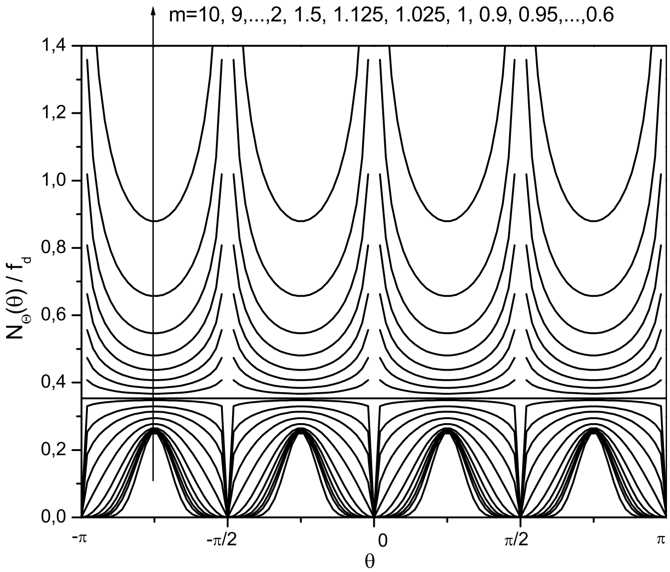

Fig. 4. Normalized PCR for several fading parameters.

of $\pi/4$, and tending to infinity at integers multiples of $\pi/2$, which is coherent with [4, Eq. 10].

VI. Conclusions

In this letter, *simple, exact, new* closed-form expressions for the generalized PCR were derived for Nakagami-*m* fading channels. Sample numerical results obtained by simulation were presented that validate the formulations developed here.

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