On the Weibull Autocorrelation and Power Spectrum Functions: Field Trials and Validation

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***Abstract*— Indoor and outdoor field trial measurements are used to validate the autocorrelation function derived in an exact manner for the Weibull fading signal. In addition, an *accurate closed-form* approximation to the power spectrum of the Weibull envelope is obtained and also validated. Comparisons are performed and an *excellent* fit to the field measurements is found.**

***Index Terms*— Field trials, power spectrum, Weibull autocorrelation function, Weibull distribution, validation.**

I. Introduction

THE multipath fading phenomenon has been characterized by several statistical models [1]. Some of them produce very accurate results, especially Rice and Nakagami-m. Another important fading model is Weibull, which was first used in problems dealing with reliability. Experimental data supporting the usefulness of the Weibull fading model for both indoor and outdoor applications have been widely reported in the literature (e.g., [2]–[5]). To the best of the authors' knowledge, the literature dealing with field measurements in Weibull fading channels has been devoted to the study of the first order statistics. Very recently [6], a simple closed-form expression for the generalized cross-moments of the Weibull distribution has been derived. The resulting correlation coefficient obtained in [6] has been written in terms of well-known physical fading parameters, consistently with the other more general joint statistics used in wireless communications, such as Rice and Nakagami-m.

In this paper, the autocorrelation function derived in [6] is validated through field measurements. In addition, an *accurate closed-form* approximation to the autocorrelation function is obtained. This is then used to obtain an *accurate closed-form* approximation to the power spectrum of the Weibull envelope, which is also validated through field measurements.

II. The Autocorrelation Function

The temporal autocorrelation function $A_R(\tau)$ of the Weibull envelope R has been obtained in [6]. For isotropic environment it is given as

$$A_R(\tau) \triangleq E[R(t)R(t+\tau)] = \hat{r}^2\,\Gamma^2\left(1+\frac{1}{\alpha}\right)\,{}_2F_1\left(-\frac{1}{\alpha},-\frac{1}{\alpha};1;J_0^2(\omega_D\tau)\right) \quad (1)$$

where $\hat{r} = \sqrt[\alpha]{E[R^\alpha]}$ is the α-root mean value of R^α, $E[\cdot]$ denotes the expectation operator, α is the Weibull parameter, $\Gamma(\cdot)$ is the Gamma function [7, Eq. 8.310.1], ${}_2F_1(\cdot)$ is the hypergeometric function [7, Eq. 9.14.1], $J_0(\cdot)$ is the Bessel function of the first kind and zeroth order [7, Eq. 8.401], and ω_D is the maximum Doppler shift given in rad/s. Using the space-time duality of the wireless channel [8], then $\omega_D\tau = 2\pi d/\lambda$, where d denotes distance, and λ is the carrier wavelength. Thus, the spatial autocorrelation function $A_R(d)$ of R is

$$A_R(d) = \hat{r}^2\,\Gamma^2\left(1+\frac{1}{\alpha}\right)\,{}_2F_1\left(-\frac{1}{\alpha},-\frac{1}{\alpha};1;J_0^2(2\pi d/\lambda)\right) \quad (2)$$

A. The moment-based α-estimator

The moments of the Weibull envelope are given as $E[R^k] = \hat{r}^k\Gamma(1+k/\alpha)$. From this [9]

$$\frac{E^i[R^j]}{E^j[R^i]} = \frac{\Gamma^i\left(1+j/\alpha\right)}{\Gamma^j\left(1+i/\alpha\right)} \quad (3)$$

For a particular case in which $i=2$ and $j=1$, (3) yields an estimator given in terms of the first and second moments. Of course, from (3), other moment-based estimators can be found.

III. The Envelope Power Spectrum

The power spectrum $S_R(\beta)$ of the Weibull fading envelope R is the Fourier transform[1] of its autocorrelation function $A_R(d)$ given by (2). Although this leads to an exact calculation, it seems that no closed-form expression can be found. In this section, an *accurate closed-form approximation* to $S_R(\beta)$ is derived. To this end, the following expansion of the hypergeometric function ${}_2F_1(\cdot)$ is used [8]

$${}_2F_1\left(-\frac{1}{\alpha},-\frac{1}{\alpha};1;J_0^2(2\pi d/\lambda)\right) = 1+\frac{1}{\alpha^2}J_0^2(2\pi d/\lambda) \quad (4)$$
$$+\frac{\left(1-\frac{1}{\alpha}\right)^2 J_0^4(2\pi d/\lambda)}{4\alpha^2}+\frac{\left(1-\frac{1}{\alpha}\right)^2\left(2-\frac{1}{\alpha}\right)^2 J_0^6(2\pi d/\lambda)}{36\alpha^2}+\ldots$$

[1]The Fourier transform $\mathcal{F}(\beta)$ of a function $f(x)$ is defined here as $\mathcal{F}(\beta) = \int_{-\infty}^{\infty} f(x)\exp(-j\beta x)dx$.

Manuscript received April 25, 2006. The associate editor coordinating the review of this letter and approving it for publication was Dr. Keith Zhang.

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Digital Object Identifier 10.1109/LCOMM.2006.060633.

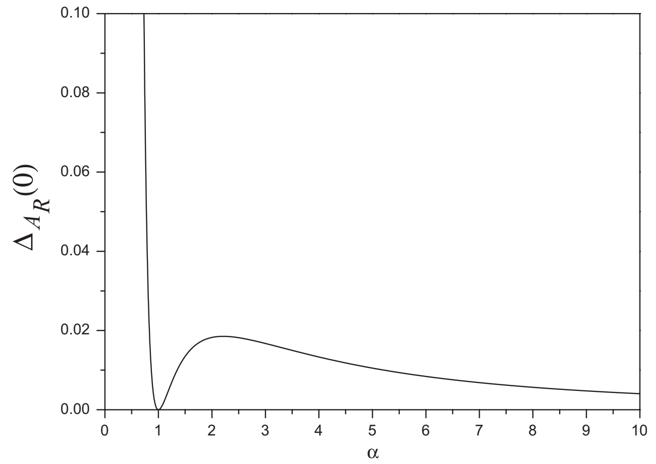

Fig. 1. Deviation of the approximated and exact Weibull autocorrelation functions for $d = 0$.

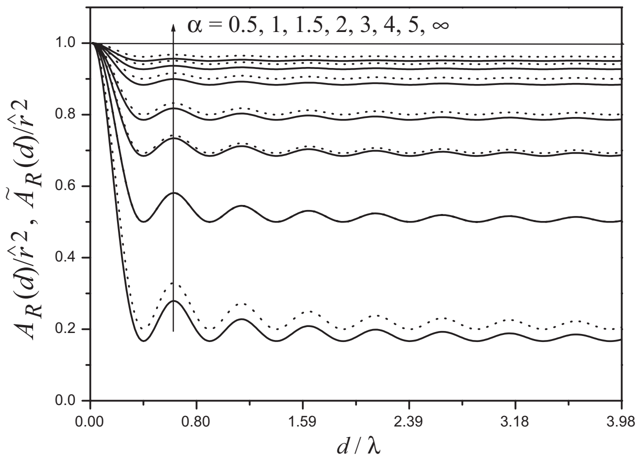

Fig. 2. Weibull normalized autocorrelation function (exact: solid; approximated: dashed).

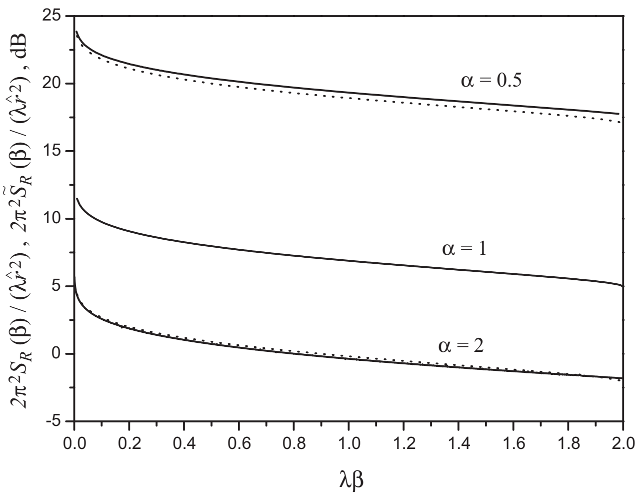

Fig. 3. Weibull normalized envelope power spectrum (exact: solid; approximated: dashed).

In (4), dropping the terms of order beyond two, the exact Weibull autocorrelation function $A_R(d)$ (2) can be approximated by $\tilde{A}_R(d)$ as

$$A_R(d) \approx \tilde{A}_R(d) = \hat{r}^2\Gamma^2\left(1+\frac{1}{\alpha}\right)\left(1+\frac{J_0^2(2\pi d/\lambda)}{\alpha^2}\right) \quad (5)$$

The maximum deviation between the exact (2) and the approximate (5) solutions occurs for $d = 0$. Defining $\Delta_{A_R}(0) = [A_R(0) - \tilde{A}_R(0)]/\hat{r}^2$, Fig. 1 plots this deviation as a function of α. Indeed, the deviation is *null* for $\alpha = 1$ and less than 1.8% for $\alpha > 1$. The maximum deviation for $\alpha > 1$ occurs at $\alpha \approx 2.21$. Also, as $\alpha \to \infty$ both (2) and (5) tend to $\hat{r}^2$ and $\Delta_{A_R}(0) = 0$. For $\alpha \to 0$, the deviation tends to infinity. However, $\alpha < 1$, which corresponds to a Nakagami-*m* parameter $m < 0.2$, is rarely found in real situations. Thus, for practical purposes ($\alpha \geq 1$, i.e. $m \geq 0.2$), the proposed approximation is indeed *excellent*.

Now, taking the Fourier transform of (5), as shall be seen, an *accurate approximation* to $S_R(\beta)$ can be written in a closed-form formula as

$$\tilde{S}_R(\beta) \approx \hat{r}^2\Gamma^2\left(1+\frac{1}{\alpha}\right) \times \left[\delta(\beta) + \frac{\lambda}{\pi^2\alpha^2}K\left(\sqrt{1-\left(\frac{\lambda\beta}{2}\right)}\right)\right] \quad (6)$$

for $|\beta| < 2/\lambda$, where $\delta(\cdot)$ is the Dirac delta function and $K(\cdot)$ is the complete elliptic integral of the first kind [7, Eq. 8.112.1]. As a check for the correctness of these results, we note that, for $\alpha = 2$ (Rayleigh condition), (5) and (6) specialize into [8, Eq. 1.3-16] and [8, Eq. 1.3-27], respectively.

A. Sample Examples

Fig. 2 illustrates how the exact and approximate autocorrelation functions of the Weibull envelope vary for different values of the parameter α. As already mentioned, for $\alpha = 1$, approximate and exact expressions are coincident. As $\alpha \to \infty$, $A_R(d) \to \hat{r}^2$, i.e., the Weibull process actually becomes a constant function. The approximation (6) to the Weibull envelope power spectrum is compared to the exact formulation (obtained by numerical integration) in Fig. 3. Both exact and approximated spectra are plotted for $\alpha = 0.5, 1$, and 2. (The dc component was omitted in these comparisons.) For $\alpha > 1$, the differences are seen to be minimal. The counterpart of the unity autocorrelation function as $\alpha \to \infty$ is a purely dc spectrum, i.e., $S_R(\beta) \to \hat{r}^2\delta(\beta)$ for $\alpha \to \infty$.

IV. Field Trials and Validation

A series of field trials was conducted at the University of Campinas (Unicamp), Brazil, in order to validate the autocorrelation function and the power spectrum of the Weibull envelope. To this end, the transmitter was placed on the rooftop of one of the buildings and the receiver traveled through the campus as well as within the buildings. The mobile reception equipment was especially assembled for this purpose. Basically, the setup consisted of a vertically polarized omnidirectional receiving antenna, a low noise amplifier, a spectrum analyzer, data acquisition apparatus, a notebook computer, and a distance transducer for carrying out the signal sampling. The transmission consisted of a CW tone at 1.8 GHz. The spectrum analyzer was set to zero span and centered

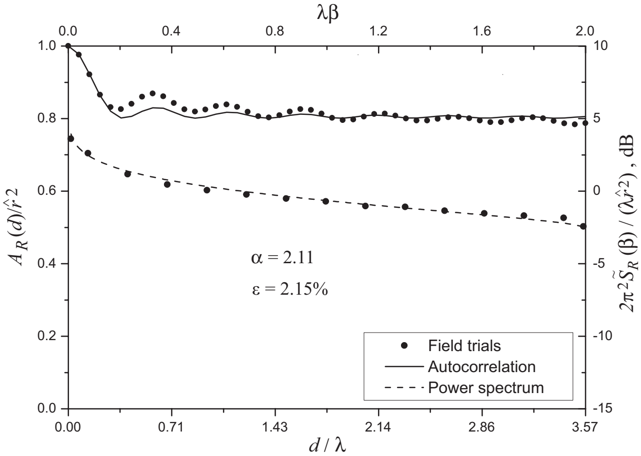

Fig. 4. Empirical versus theoretical normalized autocorrelation and normalized power spectrum functions (indoor measurements).

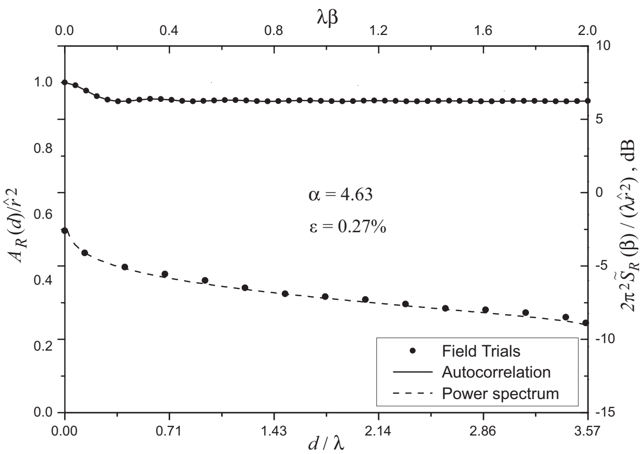

Fig. 5. Empirical versus theoretical normalized autocorrelation and normalized power spectrum functions (outdoor measurements).

at the desired frequency, and its video output used as the input of the data acquisition equipment with a sampling interval of $\lambda/14$ [10]–[12]. The local mean was estimated by the moving average method [10], with the average being conveniently taken over 450 samples symmetrically adjacent to every point. From the data collected, the long term fading was filtered out and the Weibull parameter α, as defined previously, was estimated.

The normalized empirical autocorrelation was computed according to

$$\widehat{A}_R(\Delta) = \frac{\sum_{i=1}^{N-\Delta} r_i r_{i+\Delta}}{\sum_{i=1}^{N-\Delta} r_i^2} \tag{7}$$

where r_i is the i-th sample of the amplitude sequence, N is the total number of samples (in this work $N = 10^4$), Δ is the discrete relative distance difference, and $\widehat{A}_R(\cdot)$ denotes an empirical estimate of $A_R(\cdot)$.

The empirical autocorrelation function was compared against the corresponding theoretical formula (2) and plotted as a function of d/λ with the same parameter α estimated from the experimental data. Furthermore, the mean error deviation[2], ϵ, was computed for each case. Figs. 4 and 5 show some sample plots comparing the experimental and theoretical autocorrelation data for different values of α. Observe the *excellent* fit and how the theoretical curve tends to keep track of the changes of the concavity of the empirical data. The error calculated for these curves was smaller than 2.5%.

In order to check the validity of the Weibull envelope power spectrum formulation (6), we compared it against the measured data. To this end, we used discrete Fourier transform (DFT)[3] to compute the Fourier transform of the empirical autocorrelation. Thus, the empirical envelope power spectrum S_R was computed. Figs. 4 and 5 show some sample plots comparing the experimental and theoretical power spectrum data for different values of α. Again, an *excellent* fit has been observed. The hypothesis of an isotropic scattering seems more adequate for the outdoor environment, since a somewhat large deviation between theoretical and experimental points is observed for the indoor scenario.

V. Conclusions

In this paper, we have reported the results of field trials aimed at investigating the second-order statistics of short term fading signals. An *excellent* agreement between the experimental and the theoretical data has been found. The measurements validate the autocorrelation formula derived in an exact manner in [6] for the Weibull fading signal. Moreover, an *accurate closed-form* approximation to the power spectrum of the Weibull envelope was also obtained and validated.

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[2] The mean error deviation between the measured data x_i and the theoretical value y_i is defined as $\epsilon = \frac{1}{N}\sum_{i=1}^{N}\frac{|y_i - x_i|}{x_i}$, where N is the number of points. For the present calculations, the errors were estimated for points in the interval [0,1.43λ], within which larger deviations occur.

[3] The DFT was implemented by the FFT (fast Fourier transform) algorithm.