The η-μ Joint Phase-Envelope Distribution

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***Abstract*—This letter derives a simple and closed-form expression for the phase-envelope joint distribution of the η-μ fading environment, a general fading model that includes the Hoyt and the Nakagami-m models as special cases. In the same way, the marginal distribution of the phase is also obtained in an exact closed-form expression. The phase distribution is then used in order to derive the probability density funtion (pdf) of the envelope time derivative, which, as opposed to the Nakagami-m and Rice models, is not Gaussian.**

***Index Terms*—General fading distribution, η-μ distribution.**

I. Introduction

THE η-μ distribution is a general fading distribution that includes as special cases important other distributions such as Hoyt (Nakagami-q) and Nakagami-m [1], [2]. (Therefore, the One-Sided Gaussian and Rayleigh are also special cases of it.) Its flexibility renders it suitable to better adjust to field measurement data, as demonstrated in several field measurement campaigns [2]. In particular, its tail closely follows the true statistics where other distributions fail to yield good fit. In addition, the η-μ distribution can be used to approximate the distribution of the sum of independent, nonidentical Hoyt (Nakagami-q) envelopes having arbitrary mean powers and arbitrary fading degrees. In this case, the results show that the differences between exact and approximate statistical curves are almost imperceptible [3].

As opposed to Hoyt, Rayleigh, and Rice, for which the derivation of the envelope probability density function (pdf) produced as an intermediate step the corresponding joint envelope-phase pdf, for the η-μ distribution, as well as for Nakagami-m, no information on the signal phase was provided when these distributions were proposed. Recently [4], a model for the envelope and phase of the Nakagami-m signal was presented that led to a simple joint distribution written in a closed-form manner. The corresponding phase pdf was then obtained and a compatibility with the pdfs comprised by Nakagami-m (namely, Rayleigh) or approximated by it (namely, Hoyt and Rice) was achieved.

The distribution of the phase has a great variety of applications in communications systems [5]–[9], mainly when the information is transmitted in the phase of a carrier. In special, the pdf of the phase may be useful, for instance, in determining probabilities of error for M-phase signaling over fading channels using diversity [10]. The joint envelope-phase distribution, by its turn, finds use, for instance, in the determination of higher order statistics, including level crossing rates for single or multibranch diversity scenarios [11].

The aim of this letter is to derive the η-μ envelope-phase joint pdf in an exact and closed-form manner. It is certainly desirable that compatibility with the pdfs comprised by the η-μ distribution be accomplished. In this case, the envelope-phase joint pdfs of Hoyt as well as Nakagami-m must be obtained as special cases of that derived here. From the joint envelope-phase pdf, the phase pdf is obtained, again in an exact and closed-form fashion. In addition, in order to illustrate an application of this result, the pdf of the envelope time derivative is attained.

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II. The η-μ Fading Model Revisited

The η-μ distribution is a general fading distribution that can be used to better represent the small-scale variation of the fading signal in a nonline-of-sight condition [1], [2]. As its name implies, it is written in terms of two physical parameters, namely η and μ, and it may appear in two different formats, Format 1 and Format 2, corresponding to two physical models [2]. Roughly speaking, the parameter μ is related to the number of multipath clusters in the environment, whereas the parameter η is related to the ratio of the powers (Format 1) or correlation (Format 2) between the multipath waves in the in-phase and quadrature components. These concepts are briefly revisited next.

For a fading signal with envelope R and $\hat{r}=\sqrt{E(R^2)}$ being the rms value of R, the η-μ envelope pdf is written as [1], [2]

$$f_R(r)=\frac{4\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^{\mu}}{\Gamma(\mu)H^{\mu-\frac{1}{2}}\hat{r}}\left(\frac{r}{\hat{r}}\right)^{2\mu}\exp\left[-2\mu h\left(\frac{r}{\hat{r}}\right)^2\right] \times I_{\mu-\frac{1}{2}}\left[2\mu H\left(\frac{r}{\hat{r}}\right)^2\right] \quad (1)$$

where h and H are functions of η and shall be defined next for the two different formats, $\mu > 0$ is given by $\mu=(E^2(R^2))/(2\mathrm{Var}(R^2))[1+(H/h)^2], \Gamma(\cdot)$ is the Gamma function ([12, eq. (6.1.1)]), $I_\nu[\cdot]$ is the modified Bessel function of the first kind and arbitrary order ν ([12, eq. (9.6.20)]), and $E(\cdot)$ and $\mathrm{Var}(\cdot)$ denote the expectation and variance operators, respectively. In the sequel, we present the η-μ distribution and its respective fading model for each one of the formats.

A. The η-μ Fading Model—Format 1

The fading model for the η-μ distribution (Format 1) considers a signal composed of clusters of multipath waves propagating in an non-homogeneous environment. The in-phase and quadrature components of the fading signal within each cluster are assumed to be independent from each other and to have different powers.

In Format 1, $0 < \eta < \infty$ is the scattered wave power ratio between the in-phase and quadrature components of each cluster of multipath. In such a case, $h = (2 + \eta^{-1} + \eta)/(4)$ and $H = (\eta^{-1} - \eta)/(4)$, then $H/h = (1-\eta)/(1+\eta)$. We note that within $0 < \eta \leq 1$ the envelope distribution yields the same values as for within $0 < \eta^{-1} \leq 1$, i.e., as far as the envelope is concerned, it is symmetrical around $\eta = 1$.

B. The η-μ Fading Model—Format 2

The fading model for the η-μ distribution (Format 2) considers a signal composed of clusters of multipath waves propagating in an non-homogeneous environment. The in-phase and quadrature components of the fading signal within each cluster are assumed to have identical powers and to be correlated with each other.

In Format 2, $-1 < \eta < 1$ is the correlation coefficient between the scattered wave in-phase and quadrature components of each cluster of multipath. In such a case, $h = (1)/(1 - \eta^2)$ and $H = (\eta)/(1 - \eta^2)$, then $H/h = \eta$. We note that within $0 < \eta \leq 1$ the envelope distribution yields the same values as for within $-1 < \eta \leq 0$, i.e., as far as the envelope is concerned, it is symmetrical around $\eta = 0$.

C. Format 1 and Format 2

From the above discussion, we see that $H/h = (1-\eta)/(1+\eta)$ in Format 1 and $H/h = \eta$ in Format 2. Then, it can be easily seen that one format may be obtained from another by the bilinear relation $\eta_1 = (1 - \eta_2)/(1 + \eta_2)$ or, equivalently, $\eta_2 = (1-\eta_1)/(1+\eta_1)$, where η_1 is the parameter η for Format 1 and η_2 is the parameter η for Format 2.

III. Derivation of the η-μ Joint Phase-Envelope Distribution

Let $R\exp(j\Theta)$ be the signal following the η-μ distribution, in which R represents the envelope and Θ the phase. Then, it follows that

$$R^2 = X^2 + Y^2 \tag{2}$$

$$\Theta = \arctan(Y/X) \tag{3}$$

where $X^2 = \sum_{i=1}^{2\mu} X_i^2$ and $Y^2 = \sum_{i=1}^{2\mu} Y_i^2, X_i$, and Y_i are Gaussian variates, and 2μ corresponds to the number of multipath clusters, assumed integer initially and then extended to real. In Format 1, X_i and Y_i are zero-mean mutually independent processes, i.e., $E(X_i) = E(Y_i) = 0$, and non-identical variances so that $E(X_i^2) = \eta\hat{r}^2/(2\mu(1+\eta)) = \Omega_X/2\mu$ and $E(Y_i^2) = \hat{r}^2/(2\mu(1+\eta)) = \Omega_Y/2\mu$. In Format 2, X_i and Y_i are zero-mean mutually correlated processes with $E(X_i) = E(Y_i) = 0$ and identical variances so that $E(X_i^2) = E(Y_i^2) = \hat{r}^2/4\mu = \Omega/4\mu$ and $\eta = 4\mu E(X_iY_i)/\Omega$. Departing from the correlated variates X_i and Y_i (Format 2) and making a rotation of axis, we arrive at independent in-phase and quadrature variates (Format 1-like) having power parameters respectively as $\Omega_X = ((1-\eta)\Omega)/(2)$ and $\Omega_Y = ((1+\eta)\Omega)/(2)$.

For both formats, Ω_X and Ω_Y can be expressed in the same manner as

$$\Omega_X = \frac{(h-H)\hat{r}^2}{2h} \tag{4}$$

$$\Omega_Y = \frac{(h+H)\hat{r}^2}{2h} \tag{5}$$

with h and H assuming different values for the different formats.

From (2), it is possible to write X and Y as $X = R\cos\Theta$ and $Y = R\sin\Theta$. Let $Z^2 = \sum_{i=1}^{2\mu} G_i^2$, where $Z = X, G_i = X_i$, and $\Omega_Z = \Omega_X$, or $Z = Y, G_i = Y_i$, and $\Omega_Z = \Omega_Y$, as required. Now, expressing Z as $Z = \mathrm{sgn}(Z)\times|Z|$, where $\mathrm{sgn}(\cdot)$ denotes the sign function, and performing a similar procedure as that of [4], then the pdf $f_Z(z)$ of Z is obtained as

$$f_Z(z) = \frac{\mu^\mu |z|^{2\mu-1}}{\Omega_Z^\mu \Gamma(\mu)} \exp\left(-\frac{\mu z^2}{\Omega_Z}\right), \quad -\infty < z < \infty \tag{6}$$

Knowing that, after the transformation described, X and Y are independent variates for both formats, then their joint pdf is given by $f_{X,Y}(x,y) = f_X(x) \times f_Y(y)$. It follows that

$$f_{X,Y}(x,y) = \frac{\mu^{2\mu}|x|^{2\mu-1}|y|^{2\mu-1}}{\Omega_X^\mu \Omega_Y^\mu \Gamma^2(\mu)} \times \exp\left(-\mu\left(\frac{x^2}{\Omega_X} + \frac{y^2}{\Omega_Y}\right)\right) \tag{7}$$

Using the standard statistical procedure of transformation of variates so that $f_{R,\Theta}(r,\theta) = |J| f_{X,Y}(x,y)$, where $|J| = r$ is the Jacobian of the transformation, and substituting (4) and (5) in (7), the corresponding joint pdf $f_{R,\Theta}(r,\theta)$ is given by

$$f_{R,\Theta}(r,\theta) = \frac{2\mu^{2\mu}h^{2\mu}r^{4\mu-1}|\sin(2\theta)|^{2\mu-1}}{(h^2-H^2)^\mu \hat{r}^{4\mu}\Gamma^2(\mu)} \times \exp\left(-\frac{2\mu h r^2}{\hat{r}^2(h^2-H^2)}(h + H\cos(2\theta))\right) \tag{8}$$

Note that, as opposed to the Nakagami-m case, R is not independent of Θ. Integrating (8) with respect to r, we obtain the phase pdf $f_\Theta(\theta)$ as

$$f_\Theta(\theta) = \frac{(h^2-H^2)^\mu \Gamma(2\mu)|\sin(2\theta)|^{2\mu-1}}{2^{2\mu}\Gamma^2(\mu)(h + H\cos(2\theta))^{2\mu}} \tag{9}$$

Although derived for integer values of 2μ, there are no mathematical constraints for these expressions to be used for any $\mu > 0$. It should be emphasized that (8) and (9) are general novel closed-form expressions. Note also that, in the η-μ model, envelope and phase are dependent random variables.

IV. Some Discussions

As already mentioned, the envelope pdf of the η-μ distribution is symmetrical about $\eta = 1$, for Format 1, and about $\eta = 0$, for Format 2, i.e., it is irrelevant if it is explored within one or another range of this parameter. However, this is not true for its

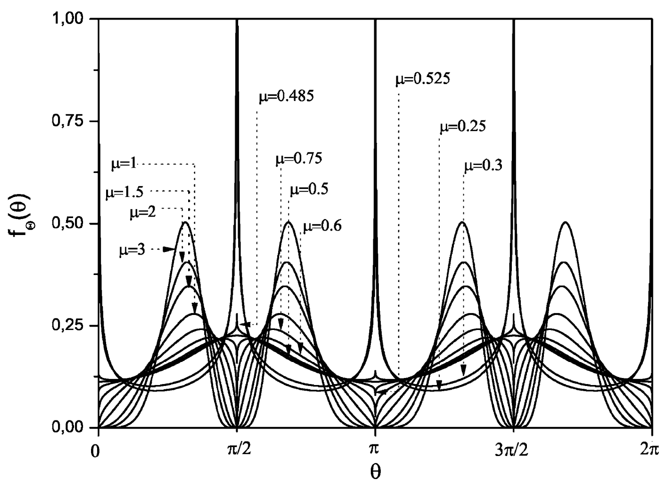

Fig. 1. pdf of the phase for η-μ fading model Format 1 ($\eta = 0.5$ and μ varying).

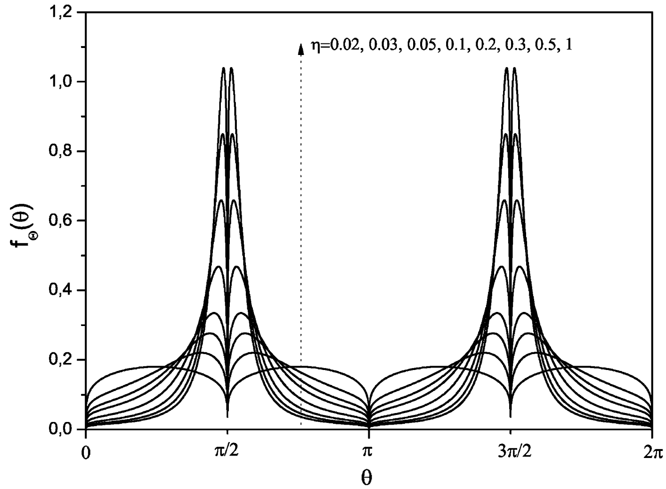

Fig. 2. pdf of the phase for η-μ fading model Format 1 ($\mu = 0.6$ and η varying).

phase pdf, in which case a shift of $\pi/2$ is observed between the curves within the respective ranges. The result in (9) reduces to that of Hoyt ([13, eq. (6.3)]) in an exact manner for $\mu = 0.5$ and for the Hoyt parameter $b = -(H/h)$. In the same way, it reduces to exactly that of Nakagami-m ([4, eq. (3)]) for $\mu = m/2$ and $(H/h) \to 0$. The case for which $(H/h) \to 1$ is of special interest. Although, this condition leads to a Nakagami-m envelope pdf, the same does not happen with the phase pdf. As can be seen from (9), except for $\theta = \pi/2$ and $\theta = 3\pi/2$ where impulses occur, the phase pdf is constantly nil throughout the whole range of θ. Its dual case is that for which $(H/h) \to -1$, when impulses occur at $\theta = \pi$ and $\theta = 2\pi$. This result is indeed expected since $(H/h) \to 1$ signifies the existence of the quadrature component only, whereas $(H/h) \to -1$ signifies the existence of the in-phase component only.

In this section, we show some plots of the η-μ phase distribution. For brevity, only Format 1 is depicted. One format may be obtained from another by the bilinear transformation, as described before. Fig. 1 illustrates the pdf of the phase for $\eta = 0.5$ and μ varying from 0.25 to 3. For values of μ smaller than 0.5, the curves are convex tending to infinity at integers multiples of $\pi/2$. For $\mu = 0.5$, the curve reaches its minimum at integers multiples of π and maximum at odd multiples of $\pi/2$. For values of μ greater than 0.5, the curves assume null value at integer multiples of $\pi/2$. Fig. 2 plots the phase pdf for $\mu = 0.6$ and η varying. Fig. 3 shows the phase pdf in polar coordinates with $\eta = 0.5$ and μ varying.

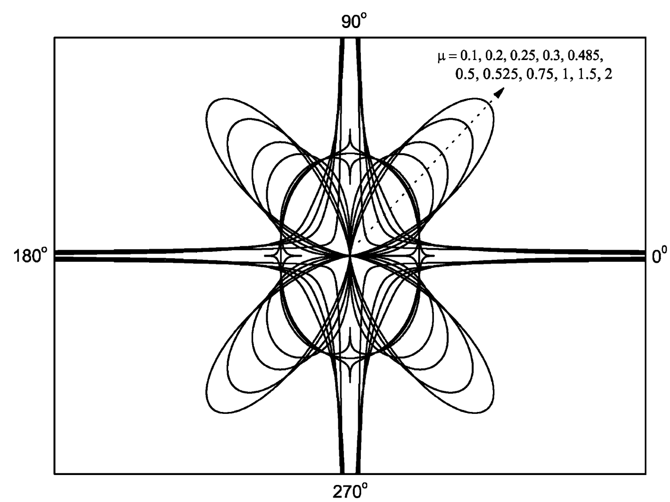

Fig. 3. pdf of the phase for η-μ fading model Format 1—polar coordinates ($\eta = 0.5$ and μ varying).

V. Further Results

As mentioned before, the distribution of the phase has a great variety of applications in communications systems. The joint envelope-phase distribution, for instance, finds use in the determination of higher order statistics, including level crossing rates for single or multibranch diversity scenarios. In this Section, a glimpse at a possible application of the result obtained here is given. In particular, the pdf $f_{\dot{R}}(\dot{r})$ of the envelope time derivative $\dot{R}$ is derived. In order to do so, we make use of the conditional pdf $f_{\dot{R}|\Theta}(\dot{r}|\theta)$ of $\dot{R}$ given Θ, as follows:

$$f_{\dot{R}}(\dot{r}) = \int_0^{2\pi} f_{\dot{R}\,|\,\Theta}(\dot{r}\,|\,\theta) f_\Theta(\theta)\, d\theta \tag{10}$$

From (2), $R\dot{R} = X\dot{X} + Y\dot{Y}$, where $\dot{X}$ and $\dot{Y}$ denote, respectively, the time derivatives of X and Y. Knowing that $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$, then

$$\dot{R} = \dot{X}\cos(\Theta) + \dot{Y}\sin(\Theta) \tag{11}$$

Because Z ($Z = X$, or $Z = Y$) is Nakagami-m distributed [4] and its time derivative is Gaussian distributed [14], then for isotropic scattering $\dot{Z}$ is zero-mean Gaussian with $E(\dot{Z}^2) = (\pi f_m)^2\, \Omega_Z/\mu$, where f_m is the maximum Doppler shift. From (11), it can be seen that $\dot{R}$, given Θ, is zero-mean Gaussian distributed with variance $E(\dot{R}^2\,|\,\Theta)$. By reducing the dependent in-phase and quadrature components of Format 2 into indepen-

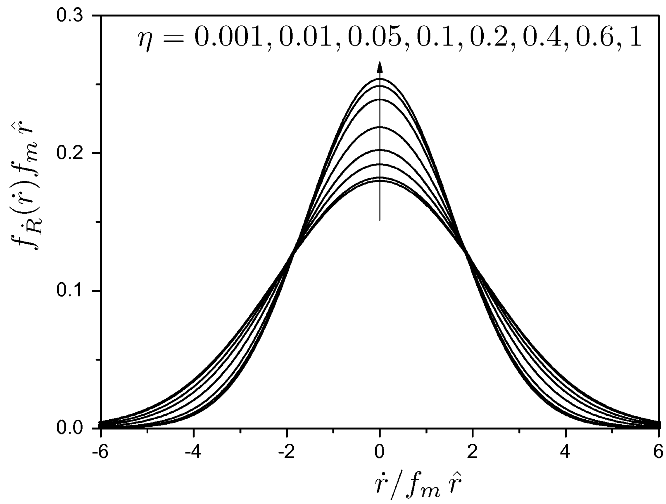

Fig. 4. pdf of the envelope time derivative for η-μ fading model Format 1 ($\mu = 2$ and η varying).

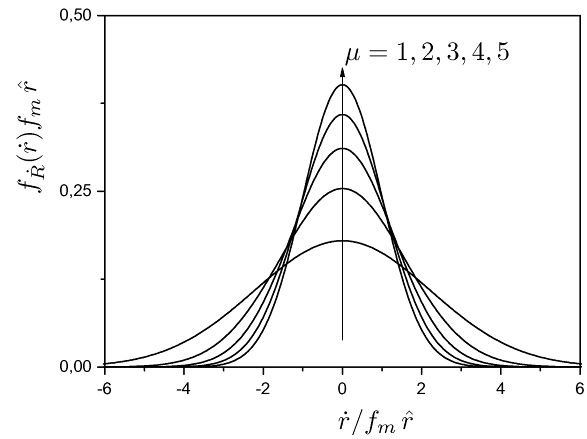

Fig. 5. pdf of the envelope time derivative for η-μ fading model Format 1 ($\eta = 1$ and μ varying).

dent in-phase and quadrature components of Format 1, as before, such a variance can be easily found as

$$E(\dot{R}^2 \,|\, \Theta) = \frac{f_m^2 \pi^2 \, \hat{r}^2}{2\mu} \left(1 - \frac{H\cos(2\theta)}{h}\right) \tag{12}$$

for both formats. By substituting (9) into (10) and knowing that $f_{\dot{R}\,|\,\Theta}(\dot{r}\,|\,\theta)$ is a zero-mean Gaussian distribution with variance as in (12), the pdf of $\dot{R}$ is obtained as in (13)

$$\begin{aligned} f_{\dot{R}}(\dot{r}) = {} & \frac{(\mu h)^{\frac{1}{2}} (h^2 - H^2)^{\mu} \Gamma(2\mu)}{4^{\mu-1} f_m \pi^{\frac{3}{2}} \hat{r} \Gamma^2(\mu)} \\ & \times \int_0^{\frac{\pi}{2}} \frac{(h + H\cos(2\theta))^{-2\mu} (\sin(2\theta))^{2\mu-1}}{\sqrt{h - H\cos(2\theta)}} \\ & \times \exp\left(\frac{-h\mu \dot{r}^2}{f_m^2 \pi^2 \hat{r}^2 (h - H\cos(2\theta))}\right) d\theta \end{aligned} \tag{13}$$

For illustration purposes, Figs. 4 and 5 plot (Format 1) the normalized pdf of $\dot{R}, f_{\dot{R}}(\dot{r}) f_m \hat{r}$, as a function of the normalized envelope time derivative, $\dot{r}/f_m \hat{r}$, for several fading conditions. Note the Gaussian-like shape of the curves, although $f_{\dot{R}}(\dot{r})$ is not Gaussian as are the cases of the Nakagami-m and Rice fading models.

Many other higher other statistics may be found with the aid of the results obtained here, but this is certainly out of the scope of this Letter.

VI. Conclusion

In this letter, the phase-envelope joint pdf as well as the phase pdf of the η-μ fading model have been obtained in an exact manner. The formulations derived here comprise those of Hoyt and Nakagami-m statistics found elsewhere in the literature. An interesting feature related to the shape of the curves is that the distribution of the phase for the range $-1 \leq (H/h) \leq 0$ is the same for the range $0 \leq (H/h) \leq 1$, but shifted by $\pi/2$. The phase distribution was then used in order to derive the pdf of the envelope time derivative.

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